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Non-Monotonicity and Contraposition

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Abstract
This paper develops a formal theory of non-monotonic consequence which differs from most extant theories in that it assumes Contraposition as a basic principle of defeasible reasoning. We define a minimal logic that combines Contraposition with three uncontroversial inference rules, and we prove some key results that characterize this logic and its possible extensions.

1 Overview

It is widely acknowledged that an adequate formal theory of defeasible reasoning must not include *Monotonicity*, the principle according to which, if β follows from α, then it also follows from $\alpha \wedge \gamma$. A consequence relation that does not have this property — and so qualifies as non-monotonic — crucially differs from the classical relation of logical consequence. The question that remains to be settled, however, is which of the properties of the latter relation should be retained in order to provide a positive characterization of defeasible reasoning.

Several attempts have been made to answer this question, which lies at the core of non-monotonic logic. In a seminal paper on the topic, Gabbay initially suggested a restricted set of fundamental properties of non-monotonic logic.[1] His proposal has then been elaborated and refined in different ways. Notably, Kraus, Lehmann, and Magidor identified a set of properties of non-monotonic systems — known as KLM logic — which included Gabbay's properties.[2] The current literature on non-monotonic logic contains a wide variety of formal theories that develop similar ideas.

There is one point, however, on which most of these theories tend to agree, and which we do not find entirely satisfactory: *Contraposition* — the principle according to which, if β follows from α, then $\neg\alpha$ follows from $\neg\beta$ — is hardly regarded as an

[1] Gabbay (1985).

[2] Kraus et al. (1990).

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essential trait of defeasible reasoning. As a result, mainstream contributions to non-monotonic logic rarely ever mention Contraposition. As far as we can see, however, no fully compelling reason has ever been provided for thinking that defeasible reasoning is non-contrapositive.

The marginality of Contraposition in the current literature can be explained essentially in two ways. In the first place, it is often assumed that there are intuitive counterexamples to Contraposition. This holds in particular for those who construe non-monotonic consequence in terms of normality or typicality in KLM style. Freund, Lehmann, and Morris use the following description to motivate their rejection of Contraposition:

> Contraposition is the property that says that, if we hold that birds typically fly we should hold also that non-flying things are typically non-birds.[3]

In the second place, it has become customary to rule out Contraposition on the basis of formal arguments that concern its interaction with other principles. In particular, as Kraus, Lehmann, and Magidor pointed out, if one rejects Monotonicity, one cannot retain both Contraposition and *Right Weakening*, the principle according to which, if β follows from α, then so does $\beta \vee \gamma$, because the conjunction of Contraposition and Right Weakening entails Monotonicity.[4]

However, neither of the two motivations seems very compelling. First of all, we find alleged counterexamples to Contraposition largely inconclusive. Although some arguments in natural language may seem to prove Contraposition invalid, plausible alternative interpretations are available in which those arguments do not instantiate Contraposition.[5] As to the example discussed in the quote above, it is not obvious that there is something wrong in the claim that non-flying things are typically non-birds. On some plausible reading of ‘typically’, this claim seems true. Moreover, and more importantly, we doubt that non-monotonic consequence is to be construed in terms of normality or typicality, even though the latter notions may play some key role in defeasible reasoning. And if the alleged counterexamples are rephrased so as to avoid any explicit reference to normality or typicality — say, replacing ‘birds typically fly’ with ‘Tweetie is a bird; therefore, Tweetie flies’, then arguably there is no intuitive violation of Contraposition.

The formal arguments that hinge on Right Weakening fare no better. The rejection of Monotonicity by itself does not imply that Contraposition should fail, unless some independent justification is provided for Right Weakening. As far as we can see, Right Weakening is in no better position than Contraposition in that respect. In fact there may be reasons for thinking that Right Weakening should *not* be part of a coherent

3 Freund, Lehmann, and Morris (1991), p. 191.

4 Kraus, Lehmann, and Magidor (1990), pp. 180–181.

5 In the literature on conditionals, alleged counterexamples to Contraposition typically involve the concessive reading of ‘if’, for example, from ‘If the US halts the bombing, North Vietnam will not agree to negotiate’ it does not seem to follow ‘If North Vietnam agrees to negotiate, the US will not have halted the bombing’ (see Stalnaker 1968). But such concessive constructions are *un*intended and unwelcome interpretations of non-monotonic consequence for our purposes: in the sense that matters here, the premise ‘The US halts the bombing’ plays no justificatory role for the conclusion ‘North Vietnam will not agree to negotiate’. This point has been made by Lycan (2001), p. 34, Bennett (2003), pp. 32 and 143–144, Gomes (2019).

account of defeasible reasoning. Although Right Weakening is very plausible when one restricts consideration to conclusive reasoning, it becomes more problematic as a rule for reasoning in general. Here we regard the analogy between consequence and conditionals potentially fruitful. A new trend that has emerged recently in the literature is to analyse conditionals in terms of inferences. Inferentialist theories of conditionals tend to converge on the invalidity of Right Weakening, for they assume that the relation of support that can obtain between the antecedent and the consequent can fail to be preserved by weakening the consequent: it may be the case that α is a reason for β without thereby being a reason for $\beta \vee \gamma$, because by weakening the conclusion the positive relevance of the premise can decrease or get lost.[6] A similar case can be made for non-monotonic consequence: arguably, $\beta \vee \gamma$ may fail to follow from α even though β does.

The dilemma between Contraposition and Right Weakening can be traced back to a famous philosophical dispute between Carnap and Hempel on inductive inference. In his early work on confirmation, Hempel endorsed the so-called Special Consequence Condition, which is an analogue of Right Weakening as it says that, if α confirms β, then α also confirms anything that logically follows from β.[7] Carnap, by contrast, suggested a probabilistic measure of relevance which encodes the idea of Contraposition but does not preserve Right Weakening. Interestingly, he elucidated the tension between his view and Hempel's by postulating two distinct explicata for inductive logic: the firmness of a statement given some evidence, and the increase in firmness of a statement as provided by some evidence.[8]

The line of thought articulated here—which is rooted in Crupi and Iacona's work on the logic of reasons—is Carnapian in this respect, as it hinges on the idea that Contraposition is an essential feature of defeasible reasoning.[9] Consider the following arguments:

(1) Sophie can read French; therefore, she can read
(2) Sophie is French; therefore, she can read French

In (1) the reason stated by the premise is conclusive, in that it rules out the falsity of the conclusion. In (2), by contrast, the reason stated by the premise is defeasible, because it does not rule out the falsity of the conclusion. In spite of this difference, however, there is an important analogy between (1) and (2), namely, that in both cases there is a plausible sense in which the premise is incompatible with the negation of the conclusion. According to Crupi and Iacona, it is crucial to this analogy that incompatibility is a symmetric relation: when α is incompatible with $\neg\beta$, $\neg\beta$ is incompatible with α. Therefore, in both cases it is plausible to expect that, if α supports β, then $\neg\beta$ supports $\neg\alpha$. For example, the arguments (3) and (4) below seem no less compelling than (1) and (2):

(3) Sophie cannot read; therefore, she cannot read French
(4) Sophie cannot read French; therefore, she is not French

6 Douven (2016); Rott (2022); Crupi and Iacona (2022a).

7 Hempel (1945).

8 Carnap (1962).

9 Crupi and Iacona (2023).

Or at least, it is easy to imagine contexts in which (1) and (3) are judged as equally compelling, and the same goes for (2) and (4). So Contraposition seems to hold as long as the context is assumed to be fixed.[10]

Insofar as validity broadly understood is definable in terms of incompatibility, a relation that we regard as fundamental, Contraposition holds in virtue of the symmetry of this relation. Thus, as far as Contraposition is concerned, there is no difference between conclusive and defeasible inference: non-monotonicity by itself does not entail non-contrapositivity.

The same point can be appreciated by considering the connection between being a reason *for* and being a reason *against*, which holds independently of the distinction between conclusive and defeasible reasons. It is plausible to assume that, when one has a reason for β, one thereby has a reason against $\neg\beta$. That is, 'α is a reason for β' seems to entail 'α is a reason against $\neg\beta$'. Moreover, it is equally plausible to assume that, if one has a reason against $\neg\beta$, then $\neg\beta$ is itself a reason against one's reason. That is, 'α is a reason against $\neg\beta$' seems to entail '$\neg\beta$ is a reason against α'. From these two assumptions we get the conclusion that 'α is a reason for β' entails '$\neg\beta$ is a reason against α'. As long as Contraposition holds, this conclusion makes perfect sense, for 'α is a reason for β' entails '$\neg\beta$ is a reason for $\neg\alpha$'.

In the following sections we develop the idea that Contraposition is an essential feature of defeasible reasoning by exploring the possibility of a non-monotonic contrapositive consequence relation. First we define a minimal logic where Contraposition features as the characteristic principle. This logic is called **E** — for 'evidential' — in line with the terminology adopted by Crupi and Iacona.[11] Then we show some interesting relations that hold in **E** between other principles that have been widely discussed in the literature on non-monotonic logic. These relations are distinctive of **E** in that they crucially depend on Contraposition. Finally, we present a quite natural extension of **E** and discuss some of its implications.

2 Minimal Evidential Consequence

Let $\mathscr{L}$ be a standard propositional language. Let $\alpha, \beta, \gamma, \ldots$ stand for arbitrary formulas of $\mathscr{L}$, and let $\vdash$ indicate derivability in a system of classical propositional logic. The consequence relation that we want to define, expressed by the symbol $\mid\!\sim$, obtains between formulas of $\mathscr{L}$. We use the notation $\alpha \mid\!\sim \beta$ to represent the statement that β can be inferred from α, and the notation $\alpha \mid\!\not\sim \beta$ to represent the statement that β cannot be inferred from α. The symbols $\mid\!\sim$ and $\mid\!\not\sim$ thus apply to ordered pairs of formulas of $\mathscr{L}$, which we will call respectively *positive argument pairs* and *negative argument pairs*. Positive and negative argument pairs behave like formulas and their negations in a classical propositional language, namely, they are meant to be contradictory.

Although having negative argument pairs along with positive argument pairs is not a widespread solution, for our purposes it will be useful to formally represent both

[10] Crupi and Iacona (2023) spell out the notion of context in terms of circumstances of evaluation as part of a formal semantic for the notion of reasons. The account of non-monotonic consequence suggested here is in line with that treatment.

[11] Crupi and Iacona (2022a, b)

the statement that β follows from α and the statement that β does not follow from α. More specifically, some inference rules that deserve consideration require this piece of notation to be spelled out. [12] To talk about argument pairs without specifying whether they are positive or negative, we will use the generic label *argument pairs*, and the standard notation $\langle \alpha, \beta \rangle$.

Definition 2.1 **E** is defined by the following rules:

$$LLE \quad \frac{\alpha \mid\sim \gamma}{\beta \mid\sim \gamma} \text{ if} \vdash \alpha \equiv \beta$$

$$RLE \quad \frac{\gamma \mid\sim \alpha}{\gamma \mid\sim \beta} \text{ if} \vdash \alpha \equiv \beta$$

$$Exp \quad \frac{\alpha \mid\sim \beta \qquad \alpha \mid\not\sim \beta}{\langle \delta, \gamma \rangle}$$

$$Con \quad \frac{\alpha \mid\sim \beta}{\neg\beta \mid\sim \neg\alpha}$$

Each of these rules has an argument pair as a conclusion, so it justifies the assertion that a certain formula can (or cannot) be inferred from another formula. The first two rules, *Left Logical Equivalence* and *Right Logical Equivalence*, warrant that, as long as α is logically equivalent to β, one can infer $\beta \mid\sim \gamma$ from $\alpha \mid\sim \gamma$, or $\gamma \mid\sim \beta$ from $\gamma \mid\sim \alpha$. These two rules are needed in order to provide a purely model-theoretic semantics for **E**. The third rule, *Explosion*, is an argument pair version of the classical principle *ex falso quodlibet*. This rule becomes necessary in presence of negative argument pairs. The fourth rule, Contraposition, is the distinctive rule of **E**. While the first three rules generally hold in standard non-monotonic systems, such as KLM logic, Contraposition does not hold in such systems.

By relying on the four rules just presented, one can derive argument pairs from sets of argument pairs. The syntactic notion of entailment that pertains to derivability so understood is defined as follows:

Definition 2.2 A set of argument pairs Γ *syntactically entails* an argument pair $\langle \alpha, \beta \rangle$ in **E** iff there is a finite sequence of argument pairs $A_1, \ldots, A_n$ such that $A_n = \langle \alpha, \beta \rangle$ and, for every argument pair A_i in the sequence, either $A_i \in \Gamma$, or A_i is obtained from argument pairs that precede it in the sequence by means of some rule of **E**.

The system **E** identifies a minimal contrapositive consequence relation. Although this system is far too simple to provide a full formal characterization of defeasible reasoning, it can serve as a technical basis to show the implications of Contraposition. In particular, we aim to investigate other principles which are or are not compatible with Con, and spell out some key equivalences that follow from Con.

[12] See Kraus, Lehmann, and Magidor (1990) for an analysis of some crucial inference rules based on negative argument pairs.

In order to pursue our goal, it is convenient to define a semantics for **E**. Due to the simplicity of **E**, no intuitive semantics comparable to the one of KLM logic seems available.[13] We consider here the following semantics inspired by neighbourhood models for weak conditional logics.[14]

Definition 2.3 A *model* is triple $\langle W, E, [\![\ ]\!]\rangle$ where:

- W is a non-empty set;
- E is a function that assigns to every subset of W a set of subsets of W satisfying the following condition, where $-a$ denotes the complement of a:

$$\text{(Con)} \qquad \text{If } b \in E(a), \text{ then } -a \in E(-b).$$

- $[\![\ ]\!]$ is a valuation function on W for the propositional variables of $\mathscr{L}$.

The elements of W are *worlds*. E is the *evidential function*. Intuitively, E assigns to every formula α the set of formulas that follow from α according to $\mid\!\sim$. The function $[\![\ ]\!]$ is extended to arbitrary propositional formulas in the usal way: $[\![\neg\alpha]\!] = -[\![\alpha]\!]$, $[\![\alpha \wedge \beta]\!] = [\![\alpha]\!] \cap [\![\beta]\!]$, $[\![\alpha \vee \beta]\!] = [\![\alpha]\!] \cup [\![\beta]\!]$, $[\![\alpha \supset \beta]\!] = -[\![\alpha]\!] \cup [\![\beta]\!]$, and $[\![\alpha \equiv \beta]\!] = ([\![\alpha]\!] \cap [\![\beta]\!]) \cup (-[\![\alpha]\!] \cap -[\![\beta]\!])$. Leaving aside (Con), the models just defined are similar to neighbourhood models for weak conditional logics, with the difference that they have a single evidential function rather than a neighbourhood function for each world of the model. The property (Con) directly matches the rule Con. In a similar way one can identify semantic properties corresponding to further principles, as we show in what follows.

Definition 2.4 For any model $M = \langle W, E, [\![\ ]\!]\rangle$, we say that M *verifies* a positive argument pair $\alpha \mid\!\sim \beta$ iff $[\![\beta]\!] \in E([\![\alpha]\!])$, and that M *verifies* a negative argument pair $\alpha \mid\!\not\sim \beta$ iff $[\![\beta]\!] \notin E([\![\alpha]\!])$.

Since inference rules are defined in terms of argument pairs, the notion of verification can be extended to inference rules:

Definition 2.5 A model M *verifies* an inference rule R iff it is not the case that M verifies the premises of R without verifying its conclusion.

A inference rule R is *sound* with respect to a class of models when it is verified by every member of the class. In the most general case, that is, when R is verified by every model, we say that the premises of R *semantically entail* its conclusion.

Definition 2.6 A set of argument pairs Γ *semantically entails* an argument pair $\langle\alpha, \beta\rangle$ iff for every model M, if M verifies every member of Γ, then M also verifies $\langle\alpha, \beta\rangle$.

To illustrate the definitions just offered, we now prove a crucial fact about $\mid\!\sim$, namely, that it does not obey Monotonicity:

[13] One can compare this situation with the case of modal logic. While normal modal logics are characterised by relational models, non-normal systems, being syntactically simpler, require a more general but somehow less intuitive neighbourhood semantics (see Chellas (1980)).

[14] Chellas (1975); Nute (1980).

$$\frac{\alpha \mathrel{|\!\sim} \gamma}{\alpha \wedge \beta \mathrel{|\!\sim} \gamma} \quad Mon$$

Theorem 2.1 *$\alpha \mathrel{|\!\sim} \beta$ does not semantically entail $\alpha \wedge \gamma \mathrel{|\!\sim} \beta$.*

Proof Let M be a model $\langle W, E, [\![\ ]\!]\rangle$, where $W = \{w_1, w_2\}$, $[\![\alpha]\!] = W$, $[\![\gamma]\!] = \{w_1\}$, $[\![\beta]\!] = \emptyset$, and the following holds:

$$\begin{aligned}
&E(\emptyset) = \mathcal{P}(W)\\
&E(\{w_1\}) = \{\{w_1\}, W\}\\
&E(\{w_2\}) = \{\{w_2\}, W\}\\
&E(W) = \{\emptyset, W\}
\end{aligned}$$

M is a model for **E** because E satisfies (Con). Moreover, since $[\![\beta]\!] = \emptyset$ and $E([\![\alpha]\!]) = \{\emptyset, W\}$, M verifies $\alpha \mathrel{|\!\sim} \beta$. However, since $E(\{w_1\}) = E([\![\alpha \wedge \gamma]\!])$ and $\emptyset \notin E(\{w_1\})$, M does not verify $\alpha \wedge \gamma \mathrel{|\!\sim} \beta$. □

As is easy to verify, in order for Monotonicity to hold, the following condition must be satisfied:

(Mon) If $a \in E(b)$ and $c \subseteq b$, then $a \in E(c)$.

But some models for **E**, such as that considered in the proof of Theorem 2.1, do not satisfy (Mon). To see that, let $a = \emptyset$, $b = W$, and $c = \{w_1\}$.

3 Soundness and Completeness of E

In order to show that the semantics just defined is adequate, we will prove that **E** is sound and complete with respect to this semantics, namely, that for any set of argument pairs Γ and any argument pair $\langle\alpha, \beta\rangle$, Γ syntactically entails $\langle\alpha, \beta\rangle$ if and only if Γ semantically entails $\langle\alpha, \beta\rangle$. Let us start with soundness:

Theorem 3.1 *If Γ syntactically entails $\langle\alpha, \beta\rangle$, Γ semantically entails $\langle\alpha, \beta\rangle$.*

Proof Given Definition 2.2, it suffices to show that each of the four rules of **E** is sound: for every model M, if M verifies the premises of the rule, then M verifies its conclusion. LCE, RCE and Exp clearly hold. Moreover, Con is warranted by (Con). □

Note that, given Theorem 3.1, Theorem 2.1 yields that $p \mathrel{|\!\sim} q$ does not syntactically entail $p \wedge r \mathrel{|\!\sim} q$. This means that Mon is not derivable from the inference rules of **E**.

In order to prove that **E** is complete, we show that every consistent set of argument pairs has a model. Consistency is defined as follows:

Definition 3.1 A set of argument pairs Γ is *consistent* if there are no $\alpha, \beta \in \mathscr{L}$ such that Γ syntactically entails both $\alpha \mathrel{|\!\sim} \beta$ and $\alpha \mathrel{|\!\not\sim} \beta$.

By using standard methods, any consistent set of argument pairs can be extended into a maximally consistent and deductively closed set of argument pairs, where maximal consistency and deductive closure are defined as follows:

Definition 3.2 A set of argument pairs Γ is *maximally consistent* iff Γ is consistent and, if $\langle\alpha, \beta\rangle \notin \Gamma$, then $\Gamma \cup \{\langle\alpha, \beta\rangle\}$ is inconsistent.

Definition 3.3 A set of argument pairs Γ is *deductively closed* iff for every $\alpha, \beta \in \mathscr{L}$, if Γ syntactically entails $\langle\alpha, \beta\rangle$, then $\langle\alpha, \beta\rangle \in \Gamma$.

A maximally consistent and deductively closed set of argument pairs Γ can be shown to have a canonical model, which is constructed by taking worlds to be maximally consistent sets of formulas of $\mathscr{L}$. Let Max be the class of all maximally consistent sets of formulas of $\mathscr{L}$, where maximal consistency is defined in the usual way.

Definition 3.4 For $\alpha \in \mathscr{L}$, let $\hat{\alpha}$ be the set $\{\Phi \in Max \mid \alpha \in \Phi\}$.

Definition 3.5 Let the *canonical base* be the pair $\langle Max, [\![\ ]\!]_c\rangle$, where for every atomic formula p and $\Phi \in Max$, $\Phi \in [\![p]\!]_c$ iff $p \in \Phi$.

From Definitions 3.4 and 3.5 we obtain what follows:

Lemma 3.2 *For $\alpha, \beta \in \mathscr{L}$, (i) $\hat{\alpha} \subseteq \hat{\beta}$ iff $\models \alpha \supset \beta$, (ii) $(\widehat{\neg\alpha}) = -\hat{\alpha}$.*

Proof (i) and (ii) immediately follow from Definition 3.4. $\square$

Lemma 3.3 *For $\alpha \in \mathscr{L}$, $\hat{\alpha} = [\![\alpha]\!]_c$.*

Proof The proof is by induction on the complexity of α, given Definitions 3.4 and 3.5. $\square$

Now the canonical model for Γ can be defined as follows:

Definition 3.6 The *canonical model* for Γ is the tuple $M_c = \langle Max, E_c, [\![\ ]\!]_c\rangle$, where $\langle Max, [\![\ ]\!]_c\rangle$ is the canonical base, and for all $a, b \subseteq Max$, $b \in E_c(a)$ iff there are $\alpha, \beta \in \mathscr{L}$ such that $a = \hat{\alpha}$, $b = \hat{\beta}$, and $\alpha \mathrel{|\!\sim} \beta \in \Gamma$.

Lemma 3.4 *Let Γ be a maximally consistent and deductively closed set of argument pairs, and let M_c be the canonical model for Γ. Then $\alpha \mathrel{|\!\sim} \beta \in \Gamma$ iff $\hat{\beta} \in E_c(\hat{\alpha})$.*

Proof Assume that $\alpha \mathrel{|\!\sim} \beta \in \Gamma$. By Definition 3.6 it follows that $\hat{\beta} \in E_c(\hat{\alpha})$. Now assume that $\hat{\beta} \in E_c(\hat{\alpha})$. Then, by Definition 3.6, there are $\gamma, \delta \in \mathscr{L}$ such that $\hat{\alpha} = \hat{\gamma}$, $\hat{\beta} = \hat{\delta}$, and $\gamma \mathrel{|\!\sim} \delta \in \Gamma$. By Lemma 3.2, (i), $\models \gamma \equiv \alpha$ and $\models \delta \equiv \beta$, then by LLE, $\alpha \mathrel{|\!\sim} \delta \in \Gamma$, and by RLE, $\alpha \mathrel{|\!\sim} \beta \in \Gamma$. $\square$

Lemma 3.5 *Let Γ be a maximally consistent and deductively closed set of argument pairs, and let M_c be the canonical model for Γ. Then (a) M_c is a model, and (b) M_c verifies $\langle\alpha, \beta\rangle$ iff $\langle\alpha, \beta\rangle \in \Gamma$.*

Proof (a) M_c satisfies Definition 2.3. For (Con), suppose that $b \in E_c(a)$. Then, by Definition 3.6, there are $\alpha, \beta \in \mathscr{L}$ such that $a = \hat{\alpha}$, $b = \hat{\beta}$, and $\alpha \mathrel{|\!\sim} \beta \in \Gamma$. By Con, $\neg\beta \mathrel{|\!\sim} \neg\alpha \in \Gamma$, thus $\widehat{\neg\alpha} \in E_c(\widehat{\neg\beta})$, where by Lemma 3.2, (ii), $\widehat{\neg\alpha} = -\hat{\alpha} = -a$ and $\widehat{\neg\beta} = -\hat{\beta} = -b$.

(b) Assume that M_c verifies $\alpha \mathrel{|\!\sim} \beta$. Then by Definition 2.4, $[\![\beta]\!]_c \in E_c([\![\alpha]\!]_c)$, hence, by Lemma 3.3, $\hat{\beta} \in E_c(\hat{\alpha})$. It follows by Lemma 3.4 that $\alpha \mathrel{|\!\sim} \beta \in \Gamma$. By reasoning in the opposite direction one gets that if $\alpha \mid\!\sim \beta \in \Gamma$, then M_c verifies $\alpha \mid\!\sim \beta$. Now assume that M_c verifies $\alpha \mid\!\not\sim \beta$. Then by Definition 2.4, $[\![\beta]\!]_c \notin E_c([\![\alpha]\!]_c)$, hence, by Lemma 3.3, $\hat{\beta} \notin E_c(\hat{\alpha})$. It follows by Lemma 3.4 that $\alpha \mid\!\sim \beta \notin \Gamma$, and consequently, by the maximality of Γ, that $\alpha \mid\!\not\sim \beta \in \Gamma$. Conversely, if one assumes that $\alpha \mid\!\not\sim \beta \in \Gamma$, by the consistency of Γ one gets that $\alpha \mid\!\sim \beta \notin \Gamma$, and therefore, given Lemmas 3.2 and 3.4, that M_c verifies $\alpha \mid\!\not\sim \beta$. □

Theorem 3.6 *If Γ semantically entails $\langle \alpha, \beta \rangle$, Γ syntactically entails $\langle \alpha, \beta \rangle$.*

Proof Assume that Γ does not syntactically entail $\langle \alpha, \beta \rangle$. This implies that Γ is consistent, for if Γ were inconsistent, by Exp it would syntactically entail $\langle \alpha, \beta \rangle$. Let Γ^* be the deductive closure of Γ, that is, the set of all argument pairs which are syntactically entailed by Γ. Note that $\Gamma \subseteq \Gamma^*$ and Γ^* is consistent. Let M be the canonical model for Γ^*. By Lemma 3.5, M verifies all argument pairs in Γ^*, hence it verifies all argument pairs in Γ. But it does not verify $\langle \alpha, \beta \rangle$ because $\langle \alpha, \beta \rangle \notin \Gamma^*$. Therefore, Γ does not semantically entail $\langle \alpha, \beta \rangle$. □

4 Monotonicity and Right Weakening

Our minimal system **E** can be extended in different ways to obtain stronger consequence relations. In order to understand how, it is crucial to acknowledge some remarkable facts concerning inference rules that have been widely debated in the literature on non-monotonic logic.

Let us start with Right Weakening:

$$\frac{\alpha \mid\!\sim \beta}{\alpha \mid\!\sim \beta \vee \gamma} \quad RW$$

Usually, non-monotonic logics include Right Weakening, as they assume it as a basic principle of defeasible reasoning. However, things are different in **E**, because Right Weakening turns out to be equivalent to Monotonicity.

Proposition 4.1 *Mon and RW are equivalent in* **E**.

Proof To see that Mon entails RW, assume that $\alpha \mid\!\sim \beta$. By Con it follows that $\neg\beta \mid\!\sim \neg\alpha$, thus by Mon, $\neg\beta \wedge \neg\gamma \mid\!\sim \neg\alpha$, and by Con, $\neg\neg\alpha \mid\!\sim \neg(\neg\beta \wedge \neg\gamma)$. Therefore, by LLE and RLE, $\alpha \mid\!\sim \beta \vee \gamma$. To see that RW entails Mon assume that $\alpha \mid\!\sim \gamma$. Then by Con, $\neg\gamma \mid\!\sim \neg\alpha$, thus by RW, $\neg\gamma \mid\!\sim \neg\alpha \vee \neg\beta$, and by Con, $\neg(\neg\alpha \vee \neg\beta) \mid\!\sim \neg\neg\gamma$. Therefore, by LLE and RLE, $\alpha \wedge \beta \mid\!\sim \gamma$. □

The equivalence between Monotonicity and Right Weakening is a distinctive feature of **E** because it holds in virtue of Contraposition. Once Contraposition is granted, Right Weakening turns out to be exactly as strong as Monotonicity.

This shows that there is another way to look at Kraus, Lehmanm and Magidor's observation that Contraposition implies Monotonicity in presence of Right Weakening.[15] While they use this fact to argue against Contraposition, in our framework the

[15] Kraus et al. (1990). Note however that Monotonicity does not imply Contraposition in KLM logic, so that the two principles are not equivalent.

same fact leads us to conclude that Monotonicity and Right Weakening constitute an upper bound which cannot be reached by any minimally interesting extension of **E**: once you get there, your logic is no longer non-monotonic. So the question naturally arises of what *can* coherently be added to **E**. In the next section we will consider some weaker analogues of Monotonicity and Right Weakening. The interesting fact that will emerge is that the same pattern of "horizontal" equivalence relations arises at the lower levels of the hierarchy.

5 Below Monotonicity and Right Weakening

Let us start by considering five weaker analogues of Monotonicity that have been widely discussed in the literature, Cautious Monotonicity, Weak Monotonicity, Disjunctive Rationality, Negation Rationality, and Rational Monotonicity[16]:

$$\frac{\alpha \mid\!\sim \gamma \qquad \alpha \mid\!\sim \beta}{\alpha \wedge \beta \mid\!\sim \gamma} \qquad CM$$

$$\frac{\alpha \mid\!\sim \gamma \qquad \alpha \mid\!\sim \alpha \wedge \beta}{\alpha \wedge \beta \mid\!\sim \gamma} \qquad WM$$

$$\frac{\alpha \vee \beta \mid\!\sim \gamma \qquad \alpha \not\mid\!\sim \gamma}{\beta \mid\!\sim \gamma} \qquad DR$$

$$\frac{\alpha \mid\!\sim \gamma \qquad \alpha \wedge \neg\beta \not\mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma} \qquad NR$$

$$\frac{\alpha \mid\!\sim \gamma \qquad \alpha \not\mid\!\sim \neg\beta}{\alpha \wedge \beta \mid\!\sim \gamma} \qquad RM$$

At least three preliminary observations can be made by simply relying on LLE and RLE. First, Monotonicity entails Cautious Monotonicity, and Cautious Monotonicity entails Weak Monotonicity.

Proposition 5.1 *Mon entails CM, and CM entails WM.*

Proof It is obvious that Mon entails CM. To see that CM entails WM, assume that $\alpha \mid\!\sim \gamma$ and $\alpha \mid\!\sim \alpha \wedge \beta$. Then by CM, $\alpha \wedge (\alpha \wedge \beta) \mid\!\sim \gamma$, thus by LLE, $\alpha \wedge \beta \mid\!\sim \gamma$. □

Second, Monotonicity entails Disjunctive Rationality, and Disjunctive Rationality entails Negation Rationality:

Proposition 5.2 *Mon entails DR, and DR entails NR.*

Proof To see that Mon entails DR, assume that $\alpha \vee \beta \mid\!\sim \gamma$. Then by Mon, $(\alpha \vee \beta) \wedge \beta \mid\!\sim \gamma$, thus by LLE, $\beta \mid\!\sim \gamma$. To see that DR entails NR, assume that $\alpha \mid\!\sim \gamma$ and $\alpha \wedge \neg\beta \not\mid\!\sim \gamma$. From the first assumption, by LLE $(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta) \mid\!\sim \gamma$. Then by DR, $\alpha \wedge \beta \mid\!\sim \gamma$. □

[16] Cautious Monotonicity is treated as a fundamental rule of non-monotonic logic in Gabbay (1985) and Kraus et al. (1990), among other works.

Third, Monotonicity entails Rational Monotonicity:

Proposition 5.3 *Mon entails RM.*

Proof Assume that $\alpha \mid\!\sim \gamma$ and $\alpha \not\mid\!\sim \neg\beta$. By Mon, the first assumption entails that $\alpha \wedge \beta \mid\!\sim \gamma$. □

Now consider the following list, which displays five weaker analogues of Right Weakening that we call Cautious Right Weakening, Weak Right Weakening, Conjunctive Rationality, Negation Right Weakening, and Rational Right Weakening:

$$\frac{\alpha \mid\!\sim \beta \qquad \gamma \mid\!\sim \beta}{\alpha \mid\!\sim \beta \vee \gamma} \quad CRW$$

$$\frac{\alpha \mid\!\sim \beta \qquad \beta \vee \gamma \mid\!\sim \beta}{\alpha \mid\!\sim \beta \vee \gamma} \quad WRW$$

$$\frac{\alpha \mid\!\sim \beta \wedge \gamma \qquad \alpha \not\mid\!\sim \beta}{\alpha \mid\!\sim \gamma} \quad CR$$

$$\frac{\alpha \mid\!\sim \beta \qquad \alpha \not\mid\!\sim \beta \vee \neg\gamma}{\alpha \mid\!\sim \beta \vee \gamma} \quad NRW$$

$$\frac{\alpha \mid\!\sim \beta \qquad \neg\gamma \not\mid\!\sim \beta}{\alpha \mid\!\sim \beta \vee \gamma} \quad RRW$$

The five rules in this second list are right counterparts of the five rules in the first list, with the same relations of decreasing strength.[17] First, Right Weakening entails Cautious Right Weakening, which entails Weak Right Weakening.

Proposition 5.4 *RW entails CRW, and CRW entails WRW.*

Proof It is obvious that RW entails CRW. To see that CRW entails WRW, assume that $\alpha \mid\!\sim \beta$ and $\beta \vee \gamma \mid\!\sim \beta$. Then by CRW, $\alpha \mid\!\sim \beta \vee (\beta \vee \gamma)$, thus by RLE, $\alpha \mid\!\sim \beta \vee \gamma$. □

Second, Right Weakening entails Conjunctive Rationality, which entails Negation Right Weakening:

Proposition 5.5 *RW entails CR, and CR entails NRW.*

Proof To see that RW entails CR, assume that $\alpha \mid\!\sim \beta \wedge \gamma$. By RW, it follows that $\alpha \mid\!\sim (\beta \wedge \gamma) \vee \gamma$, thus by RLE, $\alpha \mid\!\sim \gamma$. To see that $CR \vdash NRW$, assume that $\alpha \mid\!\sim \beta$ and $\alpha \not\mid\!\sim \beta \vee \neg\gamma$. Then by RLE, $\alpha \mid\!\sim (\beta \vee \neg\gamma) \wedge (\beta \vee \gamma)$, thus by CR, $\alpha \mid\!\sim \beta \vee \gamma$. □

Third, Right Weakening entails Rational Right Weakening:

Proposition 5.6 *RW entails RRW.*

[17] WRW and CR have straightforward counterparts in conditional logic. They are discussed respectively in Crupi and Iacona (2021) and Rott (2022).

Proof Assume that $\alpha \mid\sim \beta$ and $\neg\gamma \mid\not\sim \beta$. By RW, the first assumption entails $\alpha \mid\sim \beta \vee \gamma$. □

We now move to the implications of Contraposition on these principles. In particular, we show that, in **E**, each of the left rules in the first list is equivalent to the corresponding right rule in the second list.

Proposition 5.7 *CM and CRW are equivalent in* **E**.

Proof To see that CM entails CRW, assume that $\alpha \mid\sim \beta$ and $\gamma \mid\sim \beta$. By Con it follows that $\neg\beta \mid\sim \neg\alpha$ and $\neg\beta \mid\sim \neg\gamma$. Thus by CM, $\neg\beta \wedge \neg\gamma \mid\sim \neg\alpha$, and by Con, $\neg\neg\alpha \mid\sim \neg(\neg\beta \wedge \neg\gamma)$. Therefore, by LLE and RLE, $\alpha \mid\sim \beta \vee \gamma$. To see that CRW entails CM, assume that $\alpha \mid\sim \beta$ and $\alpha \mid\sim \gamma$. By Con it follows that $\neg\beta \mid\sim \neg\alpha$ and $\neg\gamma \mid\sim \neg\alpha$. Thus by CRW, $\neg\gamma \mid\sim \neg\alpha \vee \neg\beta$, and by Con, $\neg(\neg\alpha \vee \neg\beta) \mid\sim \neg\neg\gamma$. Therefore, by LLE and RLE, $\alpha \wedge \beta \mid\sim \gamma$. □

Proposition 5.8 *WM and WRW are equivalent in* **E**.

Proof To see that WM entails WRW, assume that $\alpha \mid\sim \beta$ and $\beta \vee \gamma \mid\sim \beta$. By Con it follows that $\neg\beta \mid\sim \neg\alpha$ and $\neg\beta \mid\sim \neg(\beta \vee \gamma)$, which by LLE yields $\neg\beta \mid\sim \neg\beta \wedge \neg\gamma$. By WM we get that $\neg\beta \wedge \neg\gamma \mid\sim \neg\alpha$, hence by Con and RLE that $\alpha \mid\sim \beta \vee \gamma$. To see that WRW entails WM, assume that $\alpha \mid\sim \gamma$ and $\alpha \mid\sim \alpha \wedge \beta$. By Con it follows that $\neg\gamma \mid\sim \neg\alpha$ and $\neg(\alpha \wedge \beta) \mid\sim \neg\alpha$, which yields $\neg\alpha \vee \neg\beta \mid\sim \neg\alpha$ by LLE. By WRW, then $\neg\gamma \mid\sim \neg\alpha \vee \neg\beta$. By Con it follows that $\neg(\neg\alpha \vee \neg\beta) \mid\sim \neg\neg\gamma$. Therefore, by LLE and RLE, $\alpha \wedge \beta \mid\sim \gamma$. □

Proposition 5.9 *DR and CR are equivalent in* **E**.

Proof To see that DR entails CR, assume that $\alpha \mid\sim \beta \wedge \gamma$ and $\alpha \mid\not\sim \beta$. The first assumption, by Con, entails that $\neg(\beta \wedge \gamma) \mid\sim \neg\alpha$, hence, by LLE, that $\neg\beta \vee \neg\gamma \mid\sim \neg\alpha$. From the second assumption we get that $\neg\beta \mid\not\sim \neg\alpha$, for if were the case that $\neg\beta \mid\sim \neg\alpha$, by Con we would get that $\neg\neg\alpha \mid\sim \neg\neg\beta$, hence by LLE and RLE that $\alpha \mid\sim \beta$. So we can apply DR and obtain that $\neg\gamma \mid\sim \neg\alpha$. By Con it follows that $\neg\neg\alpha \mid\sim \neg\neg\gamma$, hence, by LLE and RLE, that $\alpha \mid\sim \gamma$. To see that CR entails DR, assume that $\alpha \vee \beta \mid\sim \gamma$ and $\alpha \mid\not\sim \gamma$. The first assumption, by Con, entails that $\neg\gamma \mid\sim \neg(\alpha \vee \beta)$, so by RLE we get that $\neg\gamma \mid\sim \neg\alpha \wedge \neg\beta$. The second assumption, as in the case considered above, entails that $\neg\gamma \mid\not\sim \neg\alpha$. Then by CR, $\neg\gamma \mid\sim \neg\beta$, thus by Con, $\neg\neg\beta \mid\sim \neg\neg\gamma$. By LLE and RLE it follows that $\beta \mid\sim \gamma$. □

Proposition 5.10 *NR and NRW are equivalent in* **E**.

Proof To see that NR entails NRW, assume that $\alpha \mid\sim \beta$ and $\alpha \mid\not\sim \beta \vee \neg\gamma$. By Con it follows that $\neg\beta \mid\sim \neg\alpha$ and $\neg(\beta \vee \neg\gamma) \mid\not\sim \neg\alpha$, thus by LLE, $\neg\beta \wedge \neg\neg\gamma \mid\not\sim \neg\alpha$. Then by NR, $\neg\beta \wedge \neg\gamma \mid\sim \neg\alpha$, and by Con, $\neg\neg\alpha \mid\sim \neg(\neg\beta \wedge \neg\gamma)$, therefore by LLE and RLE, $\alpha \mid\sim \beta \vee \gamma$. To see that NRW entails NR, assume that $\alpha \mid\sim \gamma$ and $\alpha \wedge \neg\beta \mid\not\sim \gamma$. By Con it follows that $\neg\gamma \mid\sim \neg\alpha$ and $\neg\gamma \mid\not\sim \neg(\alpha \wedge \neg\beta)$, thus by RLE, $\neg\gamma \mid\not\sim \neg\alpha \vee \neg\neg\beta$. Then by NRW, $\neg\gamma \mid\sim \neg\alpha \vee \neg\beta$, and by Con, $\neg(\neg\alpha \vee \neg\beta) \mid\sim \neg\neg\gamma$, therefore by LLE and RLE, $\alpha \wedge \beta \mid\sim \gamma$. □

Proposition 5.11 *RM and RRW are equivalent in* **E**.

Proof To see that RM entails RRW, assume that $\alpha \mid\!\sim \beta$ and $\neg\gamma \mid\!\not\sim \beta$. By Con it follows that $\neg\beta \mid\!\sim \neg\alpha$ and $\neg\beta \mid\!\not\sim \neg\neg\gamma$. Then by RM, $\neg\beta \wedge \neg\gamma \mid\!\sim \neg\alpha$, and by Con, $\neg\neg\alpha \mid\!\sim \neg(\neg\beta \wedge \neg\gamma)$, therefore by LLE and RLE, $\alpha \mid\!\sim \beta \vee \gamma$. To see that RRW entails RM, assume that $\alpha \mid\!\sim \gamma$ and $\alpha \mid\!\not\sim \neg\beta$. By Con it follows that $\neg\gamma \mid\!\sim \neg\alpha$ and $\neg\neg\beta \mid\!\not\sim \neg\alpha$. Then by RRW, $\neg\gamma \mid\!\sim \neg\alpha \vee \neg\beta$, and by Con, $\neg(\neg\alpha \vee \neg\beta) \mid\!\sim \neg\neg\gamma$, therefore by LLE and RLE, $\alpha \wedge \beta \mid\!\sim \gamma$. □

We now prove that the weak forms of monotonicity considered in this section do not entail Mon even in presence of Contraposition. We consider to this purpose the following conditions over models, which ensure the soundness of CM, DR, and RM, respectively:

(Cm) If $b \in E(a)$ and $c \in E(a)$, then $c \in E(a \cap b)$.
(Dr) If $c \in E(a \cup b)$, then $c \in E(a)$ or $c \in E(b)$.
(Rm) If $c \in E(a)$ and $b \notin E(a)$, then $c \in E(a \cap -b)$.

Proposition 5.12 *CM,DR,RM, taken together, do not entail Mon*

Proof Let $M = \langle W, E, [\![\ ]\!]\rangle$, where $W = \{w_1, w_2\}$, $[\![\alpha]\!] = W$, $[\![\beta]\!] = \{w_2\}$, $[\![\gamma]\!] = \{w_1\}$, $E(\{w_2\}) = \{\emptyset, \{w_2\}, W\}$, and for all $a \neq \{w_2\}$, $E(a) = \mathcal{P}(W)$. It is easy to see that E satisfies (Con), as well as (Cm), (Dr), and (Rm). So M verifies CM, DR and RM. Moreover, M verifies $\alpha \mid\!\sim \gamma$ because $\{w_1\} \in E(W)$, however M does not verify $\alpha \wedge \beta \mid\!\sim \gamma$ because $\{w_1\} \notin E(\{w_2\})$. □

6 Further Principles

Now we consider some further pairs of rules which are equivalent in **E**. The first pair is formed by AND and OR.

$$\frac{\alpha \mid\!\sim \beta \qquad \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma}\ AND$$

$$\frac{\alpha \mid\!\sim \gamma \qquad \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}\ OR$$

Proposition 6.1 *AND and OR are equivalent in* **E**.

Proof To see that AND entails OR, assume that $\alpha \mid\!\sim \gamma$ and $\beta \mid\!\sim \gamma$. By Con it follows that $\neg\gamma \mid\!\sim \neg\alpha$ and $\neg\gamma \mid\!\sim \neg\beta$. Thus by AND, $\neg\gamma \mid\!\sim \neg\alpha \wedge \neg\beta$, and by RLE, $\neg\gamma \mid\!\sim \neg(\alpha \vee \beta)$. Then by Con, $\neg\neg(\alpha \vee \beta) \mid\!\sim \neg\neg\gamma$. Therefore, by LLE and RLE, $\alpha \vee \beta \mid\!\sim \gamma$. To see that OR entails AND, assume that $\alpha \mid\!\sim \beta$ and $\alpha \mid\!\sim \gamma$. By Con it follows that $\neg\beta \mid\!\sim \neg\alpha$ and $\neg\gamma \mid\!\sim \neg\alpha$. Thus by OR, $\neg\beta \vee \neg\gamma \mid\!\sim \neg\alpha$, and by LLE, $\neg(\beta \wedge \gamma) \mid\!\sim \neg\alpha$. Then, by Con, $\neg\neg\alpha \mid\!\sim \neg\neg(\beta \wedge \gamma)$. Therefore, by LLE and RLE, $\alpha \mid\!\sim \beta \wedge \gamma$. □

The second pair is formed by two rules that may be regarded as non-monotonic counterparts of Left Logical Equivalence and Right Logical Equivalence, that is, Left Conditional Equivalence and Right Conditional Equivalence:

$$\frac{\alpha \mid\sim \beta \qquad \beta \mid\sim \alpha \qquad \alpha \mid\sim \gamma}{\beta \mid\sim \gamma}\ LCE$$

$$\frac{\alpha \mid\sim \beta \qquad \beta \mid\sim \alpha \qquad \gamma \mid\sim \alpha}{\gamma \mid\sim \beta}\ RCE$$

Proposition 6.2 *LCE and RCE are equivalent in* $\mathbf{E}$.

Proof To see that LCE entails RCE, assume that $\alpha \mid\sim \beta$, $\beta \mid\sim \alpha$, and $\gamma \mid\sim \alpha$. By Con it follows that $\neg\beta \mid\sim \neg\alpha$, $\neg\alpha \mid\sim \neg\beta$, and $\neg\alpha \mid\sim \neg\gamma$. Thus by LCE, $\neg\beta \mid\sim \neg\gamma$, and by Con, $\neg\neg\gamma \mid\sim \neg\neg\beta$. Therefore, by LLE and RLE, $\gamma \mid\sim \beta$. To see that RCE entails LCE, assume that $\alpha \mid\sim \beta$, $\beta \mid\sim \alpha$, and $\alpha \mid\sim \gamma$. By Con it follows that $\neg\beta \mid\sim \neg\alpha$, $\neg\alpha \mid\sim \neg\beta$, and $\neg\gamma \mid\sim \neg\alpha$. Thus by RCE, $\neg\gamma \mid\sim \neg\beta$, and by Con, $\neg\neg\beta \mid\sim \neg\neg\gamma$. Therefore, by LLE and RLE, $\beta \mid\sim \gamma$. □

The third pair is formed by Impossible Antecedent and Necessary Consequent:

$$\frac{\alpha \mid\sim \bot}{\alpha \mid\sim \beta}\ IA$$

$$\frac{\top \mid\sim \beta}{\alpha \mid\sim \beta}\ NC$$

Proposition 6.3 *IA and NC are equivalent in* $\mathbf{E}$.

Proof To see that IA entails NC, assume that $\top \mid\sim \beta$. Then, by Con, $\neg\beta \mid\sim \neg\top$, thus by RLE, $\neg\beta \mid\sim \bot$. By IA it follows that $\neg\beta \mid\sim \neg\alpha$, and by Con, $\neg\neg\alpha \mid\sim \neg\neg\beta$. Therefore, by LLE and RLE, $\alpha \mid\sim \beta$. To see that NC entails IA, assume that $\alpha \mid\sim \bot$. Then, by Con, $\neg\bot \mid\sim \neg\alpha$, thus by LLE, $\top \mid\sim \neg\alpha$. By NC it follows that $\neg\beta \mid\sim \neg\alpha$, and by Con, $\neg\neg\alpha \mid\sim \neg\neg\beta$. Therefore, by LLE and RLE, $\alpha \mid\sim \beta$. □

Furthemore, we consider two connexive principles, Boethius Thesis and Aristotle's Second Thesis, in two versions, one unrestricted, the other restricted:

$$BT \qquad \frac{\alpha \mid\sim \beta}{\alpha \not\mid\sim \neg\beta}$$

$$RBT \qquad \frac{\alpha \mid\sim \beta}{\alpha \not\mid\sim \neg\beta}\ (\not\models \neg\alpha)$$

$$AST \qquad \frac{\alpha \mid\sim \beta}{\neg\alpha \not\mid\sim \beta}$$

$$RAST \qquad \frac{\alpha \mid\sim \beta}{\neg\alpha \not\mid\sim \beta}\ (\not\models \beta)$$

Proposition 6.4 *BT and AST are equivalent in* $\mathbf{E}$.

Proof To see that BT entails AST, assume that $\alpha \mid\sim \beta$. Then, by Con, $\neg\beta \mid\sim \neg\alpha$. By BT it follows that $\neg\beta \not\mid\sim \neg\neg\alpha$. Then by Con, $\neg\neg\neg\alpha \not\mid\sim \neg\neg\beta$. Therefore, by LLE and RLE, $\neg\alpha \not\mid\sim \beta$. To see that AST entails BT, assume that $\alpha \mid\sim \beta$. Then, by Con, $\neg\beta \mid\sim \neg\alpha$. By AST it follows that $\neg\neg\beta \not\mid\sim \neg\alpha$. Then by Con, $\neg\neg\alpha \not\mid\sim \neg\neg\neg\beta$. Therefore, by LLE and RLE, $\alpha \not\mid\sim \neg\beta$. □

Proposition 6.5 *RBT and $RAST$ are equivalent in* $\mathbf{E}$.

Proof To see that RBT entails $RAST$, assume that $\not\models \beta$ and $\alpha \mid\!\sim \beta$. Then $\not\models \neg\neg\beta$, and by Con, $\neg\beta \mid\!\sim \neg\alpha$. By RBT it follows that $\neg\beta \mid\!\not\sim \neg\neg\alpha$. Then by Con, $\neg\neg\neg\alpha \mid\!\not\sim \neg\neg\beta$. Therefore, by LLE and RLE, $\neg\alpha \mid\!\not\sim \beta$. To see that $RAST$ entails RBT, assume that $\not\models \neg\alpha$ and $\alpha \mid\!\sim \beta$. Then by Con, $\neg\beta \mid\!\sim \neg\alpha$, thus by $RAST$, $\neg\neg\beta \mid\!\not\sim \neg\alpha$. Then by Con, $\neg\neg\alpha \mid\!\not\sim \neg\neg\neg\beta$. Therefore, by LLE and RLE, $\alpha \mid\!\not\sim \neg\beta$. □

Finally, we show that the following two principles, that can be respectively seen as the hard and the light direction of the deduction theorem, are equivalent in $\mathbf{E}$.

$$\frac{\alpha \wedge \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \rightarrow \gamma} \quad HDT$$

$$\frac{\alpha \mid\!\sim \beta \rightarrow \gamma}{\alpha \wedge \beta \mid\!\sim \gamma} \quad LDT$$

Proposition 6.6 *HDT and LDT are equivalent in* $\mathbf{E}$.

Proof To see that HDT entails LDT, assume that $\alpha \mid\!\sim \beta \rightarrow \gamma$. Then, by Con, $\neg(\beta \rightarrow \gamma) \mid\!\sim \neg\alpha$, thus by LLE, $\neg\gamma \wedge \beta \mid\!\sim \neg\alpha$. By HDT it follows that $\neg\gamma \mid\!\sim \beta \rightarrow \neg\alpha$, and by Con, $\neg(\beta \rightarrow \neg\alpha) \mid\!\sim \neg\neg\gamma$. Therefore, by LLE and RLE, $\alpha \wedge \beta \mid\!\sim \gamma$. To see that LDT entails HDT, assume that $\alpha \wedge \beta \mid\!\sim \gamma$. Then, by Con, $\neg\gamma \mid\!\sim \neg(\alpha \wedge \beta)$, thus by RLE, $\neg\gamma \mid\!\sim \beta \rightarrow \neg\alpha$. By LDT it follows that $\neg\gamma \wedge \beta \mid\!\sim \neg\alpha$, and by Con, $\neg\neg\alpha \mid\!\sim \neg(\neg\gamma \wedge \beta)$. Therefore, by LLE and RLE, $\alpha \mid\!\sim \beta \rightarrow \gamma$. □

7 Supraclassicality

Sections 4, 5 and 6 show a pattern of equivalences that hold in virtue of Contraposition. For each of the pairs of principles considered above, an extension of $\mathbf{E}$ includes one member of the pair if and only if it includes the other. Now we will consider the first and perhaps the most natural extension of $\mathbf{E}$, which is obtained by adding *Supraclassicality* to the four rules considered:

$$SC \quad \frac{}{\alpha \mid\!\sim \beta} \text{ if } \alpha \models \beta$$

Supraclassicality guarantees that $\mid\!\sim$ is a generalization of $\models$, the classical relation of logical consequence. On the assumption that $\models$ provides an adequate formal representation of conclusive reasoning, this principle expresses the idea that whenever β conclusively follows from α, it defeasibly follows from α.

At the semantic level, Supraclassicality corresponds to the following conditions on E:

(Sc) If $a \subseteq b$, then $b \in E(a)$

So the system obtained by adding Supraclassicality to $\mathbf{E}$ — we will call it **ESC** — is characterized by the class of models that satisfy both (Con) and (Sc).

ESC has some remarkable properties which concern two well known principles, *Cut* and *Transitivity*:

$$\frac{\alpha \mid\sim \beta \qquad \alpha \wedge \beta \mid\sim \gamma}{\alpha \mid\sim \gamma} \quad Cut$$

$$\frac{\alpha \mid\sim \beta \qquad \beta \mid\sim \gamma}{\alpha \mid\sim \gamma} \quad Trans$$

Cut is often regarded as uncontentious and therefore is usually listed among the basic principles of defeasible reasoning. In **ESC** things are quite different, however, because Cut turns out to be very strong. To see why, three facts must be noted. First, Cut entails Right Weakening:

Proposition 7.1 *Cut entails RW in* **ESC**.

Proof Assume that $\alpha \mid\sim \beta$. Since $\alpha \wedge \beta \models \beta \vee \gamma$, by SC we get that $\alpha \wedge \beta \mid\sim \beta \vee \gamma$. By Cut it follows that $\alpha \mid\sim \beta \vee \gamma$. □

Second, just as in **E**, Cut is equivalent to the following rule, call it Cut$_{or}$:

$$\frac{\alpha \mid\sim \beta \vee \gamma \qquad \beta \mid\sim \gamma}{\alpha \mid\sim \gamma} \quad Cut_{or}$$

Proposition 7.2 *Cut and Cut_{or} are equivalent in* **ESC**.

Proof To see that Cut entails Cut_{or}, assume that $\alpha \mid\sim \beta \vee \gamma$ and $\beta \mid\sim \gamma$. By Con it follows that $\neg(\beta \vee \gamma) \mid\sim \neg\alpha$ and $\neg\gamma \mid\sim \neg\beta$. Thus by LLE, $\neg\beta \wedge \neg\gamma \mid\sim \neg\alpha$, and by Cut, $\neg\gamma \mid\sim \neg\alpha$. Then by Con, $\neg\neg\alpha \mid\sim \neg\neg\gamma$. Therefore, by LLE and RLE, $\alpha \mid\sim \gamma$. To see that Cut_{or} entails Cut, assume that $\alpha \mid\sim \beta$ and $\alpha \wedge \beta \mid\sim \gamma$. By Con it follows that $\neg\beta \mid\sim \neg\alpha$ and $\neg\gamma \mid\sim \neg(\alpha \wedge \beta)$, thus by LLE, $\neg\gamma \mid\sim \neg\alpha \vee \neg\beta$. Then by Cut_{or}, $\neg\gamma \mid\sim \neg\alpha$, and by Con, $\neg\neg\alpha \mid\sim \neg\neg\gamma$. Therefore, by LLE and RLE, $\alpha \mid\sim \gamma$. □

Third, Cut_{or} entails Monotonicity:

Proposition 7.3 *Cut_{or} entails Mon in* **ESC**.

Proof Assume that $\alpha \mid\sim \gamma$. Since $\alpha \wedge \beta \vdash \alpha \vee \gamma$, by SC we get that $\alpha \wedge \beta \mid\sim \alpha \vee \gamma$. By Cut_{or} it follows that $\alpha \wedge \beta \mid\sim \gamma$. □

Now consider Transitivity. Like Cut, in **ESC** this rule entails Monotonicity and Right Weakening:

Proposition 7.4 *$Trans$ entails Mon in* **ESC**.

Proof Assume that $\alpha \mid\sim \beta$. Since $\alpha \wedge \gamma \vdash \alpha$, by SC $\alpha \wedge \gamma \mid\sim \alpha$. By $Trans$ we get that $\alpha \wedge \gamma \mid\sim \beta$. □

Proposition 7.5 *$Trans$ entails RW in* **ESC**.

Proof Assume that $\alpha \mid\sim \beta$. Since $\beta \vdash \beta \vee \gamma$, by SC $\beta \mid\sim \beta \vee \gamma$. By $Trans$ we get that $\alpha \mid\sim \beta \vee \gamma$. □

At the semantic level, the entailment from Transitivity to Monotonicity turns out clear if one thinks that Transitivity requires the following condition:

(Trans) If $a \in E(b)$ and $b \in E(c)$, then $a \in E(c)$.

To see that (Trans) entails (Mon), assume that (Trans) is satisfied and suppose that $a \in E(b)$ and $c \subseteq b$. By (Sc), the second conjunct entails that $b \in E(c)$, so $a \in E(c)$ by (Trans).

Although Transitivity entails Monotonicity and Right Weakening, it is weaker than Cut, in that it is entailed by Cut without entailing it.

Proposition 7.6 *Cut entails Trans in* **ESC**.

Proof Assume that $\alpha \mid\!\sim \beta$ and $\beta \mid\!\sim \gamma$. By Mon, which follows from Cut, from the second assumption we get that $\alpha \wedge \beta \mid\!\sim \gamma$. Therefore, by Cut $\alpha \mid\!\sim \gamma$. □

Proposition 7.7 *Trans does not entail Cut in* **ESC**.

Proof Let $M = \langle W, E, [\![\ ]\!]\rangle$, where $W = \{w_1, w_2, w_3\}$, $[\![\alpha]\!] = \{w_1\}$, $[\![\beta]\!] = \{w_2\}$, $[\![\gamma]\!] = \{w_3\}$, and the following holds:

$$
\begin{aligned}
&E(\emptyset) = \mathcal{P}(W)\\
&E(\{w_1\}) = \{\{w_1\}, \{w_2\}, \{w_1, w_2\}, \{w_1, w_3\}, W\}\\
&E(\{w_2\}) = \{\{w_2\}, \{w_1, w_2\}, \{w_2, w_3\}, W\}\\
&E(\{w_3\}) = \{\{w_3\}, \{w_1, w_3\}, \{w_2, w_3\}, W\}\\
&E(\{w_1, w_2\}) = \{\{w_1, w_2\}, \{\{w_2, w_3\}, W\}\\
&E(\{w_1, w_3\}) = \{\{w_1, w_3\}, \{w_2, w_3\}, W\}\\
&E(\{w_2, w_3\}) = \{\{w_2, w_3\}, W\}\\
&E(W) = \{W\}
\end{aligned}
$$

M is a model for **ESC** because E satisfies both (Con) and (Sc). E also satisfies (Trans), so M verifies $Trans$. Since $[\![\beta]\!] = \{w_2\}$, $E(\{w_1\}) = E([\![\alpha]\!])$, and $\{w_2\} \in E(\{w_1\})$, we get that M verifies $\alpha \mid\!\sim \beta$. Since $[\![\gamma]\!] = \{w_3\}$, $E(\emptyset) = E([\![\alpha \wedge \beta]\!])$, and $\{w_3\} \in E(\emptyset)$, we get that M verifies $\alpha \wedge \beta \mid\!\sim \gamma$. However, M does not verify $\alpha \mid\!\sim \gamma$ because $\{w_3\} \notin E(\{w_1\})$. □

Propositions 7.6 and 7.7 show that, although Cut and Transitivity are both strong enough to entail Monotonicity and Right Weakening, Cut occupies a higher position in the scale of strength.

To this it can be added that Transitivity is stronger than Monotonicity:

Proposition 7.8 *Mon does not entail Trans in* **ESC**.

Proof Let $M = \langle W, E, [\![\ ]\!]\rangle$, where $W = \{w_1, w_2\}$, $[\![\alpha]\!] = \{w_1\}$, $[\![\beta]\!] = \{w_2\}$, $[\![\gamma]\!] = \emptyset$, and the following holds:

$$
\begin{aligned}
&E(\emptyset) = E(\{w_2\}) = \mathcal{P}(W)\\
&E(\{w_1\}) = \{\{w_1\}, \{w_2\}, W\}\\
&E(W) = \{\{w_1\}, W\}
\end{aligned}
$$

M is a model for **ESC** because E satisfies both (Con) and (Sc). Moreover, E satisfies (Mon), so M verifies Mon. Furthermore, $[\![\beta]\!] \in E([\![\alpha]\!])$ and $[\![\gamma]\!] \in E([\![\beta]\!])$, thus M verifies $\alpha \mathrel{|\!\sim} \beta$ and $\beta \mathrel{|\!\sim} \gamma$. However, $[\![\gamma]\!] \notin E([\![\alpha]\!])$, therefore M does not verify $\alpha \mathrel{|\!\sim} \gamma$. □

Since Monotonicity and Right Weakening are equivalent in **ESC**, Proposition 7.8 shows that Transitivity is stronger than Right Weakening as well. This means that, starting from the top and going down, we have Cut, then Transitivity, and then Monotonicity and Right Weakening. The last two principles lie at the same level, in that they are tied by an "horizontal" equivalence relation. Figure 1 displays the whole hierarchy that we find in **ESC**.

One last remark on **ESC** concerns the connexive principles. Supraclassicality is inconsistent with the unrestricted version of Boethius Thesis and Aristotle's Second Thesis, because it entails that $\alpha \mathrel{|\!\sim} \beta$ whenever α is contradictory or β is provable. This means that only the restricted version of Boethius Thesis and Aristotle's Second Thesis can find place in **ESC** or in its extensions. In order to validate the unrestricted version of the connexive principles, one needs some extension of **E** that is not supraclassical.

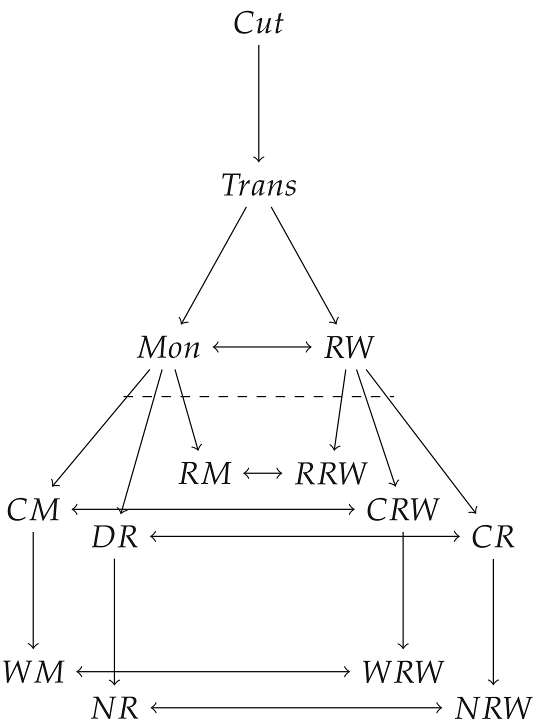

Fig. 1 Inferential relationships among logical principles

8 Conclusive Remarks

The foregoing sections suggest that, from a theoretical point of view, the combination of Non-monotonicity and Contraposition is legitimate as a starting point for the generalization of a classical consequence relation. Our preliminary results indicate that such perspective can be developed at the formal level, with potentially significant results.

This perspective opens up a number of new questions that would deserve careful consideration. A prominent issue concerns the strengthening of the minimal system **E** beyond **ESC**. How high can one go up in the scale of weakened monotonicity principles in Fig. 1 while retaining the failure of full monotonicity? To what extent can a set of weak monotonicity principles be integrated with other plausible rules such as AND, Left Conditional Equivalence or Aristotle's Second Thesis?

A comparison with recent work in conditional logic can provide much insight in this connection. In particular, there is at least one well defined non-monotonic and contrapositive conditional that provides a counterpart of an extension of **ESC**, the evidential conditional defined by Crupi and Iacona.[18] The modal semantics of the evidential conditional generates a pattern of principles that we regard as interesting: for example, it validates Cautious Monotonicity and AND, but not Rational Monotonicity and Left Conditional Equivalence.

Surely, several routes depart from the basic facts reported above, and most of the technical issues that might arise still remain to be addressed. But at least we have outlined a framework for future work. The main purpose of the paper is to show how Contraposition can serve as a legitimate cornerstone for non-monotonic logic. Once this choice is adopted, a whole structural network of logical principles is disrupted and reshaped. Largely influential rules, such as Cut, loose their privileged status, whereas other principles that are traditionally regarded as independent become pairwise coupled in a new way.

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CORRECTION

Correction: Non-Monotonicity and Contraposition

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In the original published article, the Proposition 7.1 was incorrect. The correct and incorrect version of Proposition 7.1 are provided in this correction.

Incorrect version:

Proposition 7.1 *Cut entails RW in* **ESC.**

Proof Assume that $\alpha \mid\!\sim \beta$. Since $\alpha \wedge\beta \mid\_ \beta \vee\gamma$, by *SC* we get that $\alpha \wedge\beta \mid\!\sim \beta \vee\gamma$. By *Cut* it follows that $\alpha \mid\!\sim \beta \vee \gamma$. □

Second, just as in **E**, Cut is equivalent to the following rule, call it Cut*or*:
$E(\emptyset) = P(W)$
$E(\{w1\}) = \{\{w1\}, \{w2\}, \{w1,w2\}, \{w1,w3\}, W\}$
$E(\{w2\}) = \{\{w2\}, \{w1,w2\}, \{w2,w3\}, W\}$
$E(\{w3\}) = \{\{w3\}, \{w1,w3\}, \{w2,w3\}, W\}$
$E(\{w1,w2\}) = \{\{w1,w2\}, \{\{w2,w3\}, W\}.$
$E(\{w1,w3\}) = \{\{w1,w3\}, \{w2,w3\}, W\}$
$E(\{w2,w3\}) = \{\{w2,w3\}, W\}$
$E(W) = \{W\}$

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Correct version:

Proposition 7.1 *Cut entails RW in* **ESC**.

Proof Assume that $\alpha \mid\!\sim \beta$. Since $\alpha \wedge \beta \models \beta \vee \gamma$, by SC we get that $\alpha \wedge \beta \mid\!\sim \beta \vee \gamma$. By Cut it follows that $\alpha \mid\!\sim \beta \vee \gamma$. □

Second, just as in E, Cut is equivalent to the following rule, call it Cut$_{or}$:

$$\frac{\alpha \mid\!\sim \beta \vee \gamma \qquad \beta \mid\!\sim \gamma}{\alpha \mid\!\sim \gamma} \quad Cut_{or}$$

The original article has been corrected.

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