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Chapter 23
Generalized Confirmation and Relevance Measures

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Abstract The main point of the paper is to show how popular probabilistic measures of incremental confirmation and statistical relevance with qualitatively different features can be embedded smoothly in generalized parametric families. In particular, I will show that the probability difference, log probability ratio, log likelihood ratio, odds difference, so-called improbability difference, and Gaifman's measures of confirmation can all be subsumed within a convenient biparametric continuum. One intermediate step of this project may have interest on its own, as it provides a unified representation of graded belief of which both probabilities and odds are special cases.

Keywords Inductive confirmation • Evidential support • Probabilistic relevance • Odds • Generalized logarithm

23.1 Introduction

A high level of troponin in the blood indicates a diagnosis of myocardial infarction. A matching DNA profile suggests that a suspect murderer may in fact be guilty. And the detection of the Higgs boson increased the experimental evidence in favor of so-called standard model of particle physics. In contemporary epistemology and philosophy of science, the general notion of *confirmation* or *evidential support* is often employed to interpret cases of all these different kinds.

Relying on a probabilistic account of graded credences, this idea can be characterized in a rather effective and elegant way. Consider a logical language $\boldsymbol{L}$ (finite, for simplicity), the subset $\boldsymbol{L_C}$ of its consistent formulae, and the set $\boldsymbol{P}$

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of all *regular* probability functions defined over $\boldsymbol{L}$.[1] The notion of (incremental) *confirmation* or *evidential support* given by a piece of evidence e to a hypothesis h ($h,e \in \boldsymbol{L_C}$) can be plausibly represented by a function $C(h,e)$: $\{\boldsymbol{L_C} \times \boldsymbol{L_C} \times \boldsymbol{P}\} \to \Re$, provided that some basic conditions are satisfied. The appropriate conditions are natural and compelling. On the one hand, for any fixed hypothesis h, the final probability and confirmation should always move in the same direction in the light of data, that is, for any $h,e,f \in \boldsymbol{L_C}$ and any $P \in \boldsymbol{P}$, $C(h,e) \gtreqless C(h,f)$ if and only if $P(h|e) \gtreqless P(h|f)$. Moreover, any hypothesis should be equally "non-confirmed" by empty evidence, i.e., a tautology $\top$, so for any h, $k \in \boldsymbol{L_C}$ and any $P \in \boldsymbol{P}$, $C(h,\top) = C(k,\top)$. These basic conditions are virtually sufficient to imply that there exists a fixed threshold value $C(\top,\top)$ (often set at 0) such that for any $h,e \in \boldsymbol{L_C}$ and any $P \in \boldsymbol{P}$, $C(h,e) \gtreqless C(\top,\top)$ if and only if $P(h|e) \gtreqless P(h)$. The latter, in turn, is the key and standard idea of the qualitative distinction between confirmation, neutrality, and disconfirmation, respectively: the evidence considered, e, confirms/is neutral for/disconfirms the hypothesis at issue, h, just in case the occurrence of e increases/leaves untouched/decreases the initial credibility of h (see Crupi 2015).

Specific quantitative measures of confirmation are known to be many and diverse (see, e.g., Brössel 2013; Glass 2013; Roche and Shogenji 2014), but the most popular options can be generated by a combination of three simple steps. (i) First, one can choose between two major ways to represent the credibility of a statement x, i.e., as the simple probability of x, $P(x)$, or as the *odds* for x and against not-x, $O(x) = P(x)/P(\neg x)$. (*Conditional* odds are as expected: $O(x|y) = P(x|y)/P(\neg x|y)$.) Probabilities and odds are interdefinable but not identical. According to a useful metaphor by Joyce (2004), "the difference between 'probability talk' and 'odds talk' corresponds to the difference between saying 'we are two thirds of the way there' and saying 'we have gone twice as far as we have yet to go'". (ii) Second, one can convey confirmation from e to h *directly*, as it were, by the increase in the credibility of h provided by e. Or one can do that *indirectly*, i.e., by the decrease of the credibility of the negation, $\neg h$, given e. (iii) Finally, two distinct functional forms are canonical to formalize how the relevant representation of posterior credibility departs from the prior, namely, the simple algebraic difference or the logarithm of the ratio (both of which conveniently yield 0 as a neutrality value).

In the probability formalism, the direct and indirect difference collapse on one single measure:

$$D(h,e) = P(h|e) - P(h) = P(\neg h) - P(\neg h|e)$$

[1] A *regular* probability function never assigns probability 0 to a statement unless it expresses a logical falsehood (i.e., for any $\alpha \in \boldsymbol{L_C}$, $P(\alpha) > 0$). Regularity can be motivated as a way to represent credences that are non-dogmatic as concerns $\boldsymbol{L_C}$ (see Howson 2000, p. 70). It is known to be a convenient but not entirely innocent assumption (see Festa 1999; Kuipers 2000 for discussion; also see Pruss 2013).

Originally proposed by Carnap (1950/1962), $D(h,e)$ is a natural and widespread way to quantify confirmation (Milne 2012). One does not have the same kind of convergence, however, if the log ratio is employed as a functional form. In this case, the direct and indirect forms generate, respectively:

$$R(h,e) = \ln\left[\frac{P(h|e)}{P(h)}\right]$$

$$G(h,e) = \ln\left[\frac{P(\neg h)}{P(\neg h|e)}\right]$$

Once forcefully advocated by Milne (1996), $R(h,e)$ can be seen as conveying key tenets of so-called "likelihoodist" position about evidential reasoning, as suggested by Fitelson (2007, p. 478) (see Royall 1997 for a classical statement of likelihoodism, and Chandler 2013 and Sober 1990 for consonant arguments and inclinations; also see Iranzo and Martínez de Lejarza 2012). Measures ordinally equivalent to $G(h,e)$, in turn, have been suggested and discussed by Gaifman (1979), Rips (2001), and Crupi and Tentori (2013, 2014).[2]

In the odds formalism, the direct and indirect difference measures are *not* equivalent:

$$OD(h,e) = O(h|e) - O(h) = \frac{1}{P(\neg h|e)} - \frac{1}{P(\neg h)}$$

$$ID(h,e) = O(\neg h) - O(\neg h|e) = \frac{1}{P(h)} - \frac{1}{P(h|e)}$$

The odds difference measure $OD(h,e)$ appears in Hájek and Joyce (2008, p. 122), while a thorough discussion of $ID(h,e)$ (labelled "improbability difference") has been recently provided by Festa and Cevolani (2016). Finally, when the log ratio form is applied to the odds formalism, the direct and indirect measurements do collapse on the last, highly influential element of our list, the (log) *odds ratio* measure, also equivalent to so-called (log) likelihood ratio (Good 1950; Heckerman 1988; Fitelson 2001; Park 2014):

$$OR(h,e) = \ln\left[\frac{O(h|e)}{O(h)}\right] = \ln\left[\frac{O(\neg h)}{O(\neg h|e)}\right] = \ln\left[\frac{P(e|h)}{P(e|\neg h)}\right]$$

We therefore have six popular and non-equivalent measures of incremental confirmation arising from a general scheme (see Table 23.1 for a summary). They have been shown elsewhere to exhibit diverging properties of theoretical significance (Brössel 2013; Crupi et al. 2007, 2010; Festa and Cevolani 2016). In

[2]Two ordinally equivalent measures C and C^* are such that for any $h,k,e,f \in \boldsymbol{L_c}$ and any $P \in \boldsymbol{P}$, $C(h,e) \gtreqless C(k,f)$ if and only if $C^*(h,e) \gtreqless C^*(k,f)$.

Table 23.1 A generating schema for popular measures of incremental confirmation or evidential support

probability			**odds**			
direct *vs.* inderect assessment			**direct *vs.*inderect assessment**			
$P(h\|e)$ *vs.* P(h)	$P(\neg h)$ *vs.* $P\neg(h\|e)$		$O(h\|e)$ *vs.* $O(h)$		$O(\neg h)$ *vs.* $O(\neg h\|e)$	
difference *vs.* log ratio	**difference *vs.* log ratio**		**difference *vs.* log ratio**		**difference *vs.* log ratio**	
$D(h,e)$ $P(h\|e) - P(h)$	$R(h,e)$ $\text{In}[P(h\|e)/P(h)]$	$D(h,e)$ again	$G(h,e)$ $\text{In}[P(\neg h)/P(\neg h\|e)]$	$OD(h,e)$ $O(h\|e) - O(h)$	$OR(h,e)$ $\text{In}[O(h\|e)/O(h)]$	$OR(h,e)$ again

the present contribution, my aim is solely one of theoretical unification. The main contribution will be the definition of a bi-parametric continuum of confirmation measures by which *all* of the classical options mentioned can be recovered as special cases.

23.2 Generalized Confirmation Measures

The main technical tool to achieve the parametric generalization of incremental confirmation measures that we look for is the following function ($x > 0$):

$$ln_r(x) = \frac{x^r - 1}{r}$$

Functions such as ln_r are often called generalized logarithms, because the natural logarithm, $ln(x)$, arises as a special case in the limit (when $r \to 0$). This fundamental property can be derived as follows. We posit $x = 1{-}y$ and first consider $x \leq 1$, so that $|{-}y| < 1$ (recall from above that x is strictly positive). Then we have:

$$\lim_{r\to 0}\{ln_r(x)\} = \lim_{t\to 0}\{ln_r(1-y)\} = \lim_{r\to 0}\left\{\frac{1}{r}\left[(1-y)^r - 1\right]\right\}$$

By the binomial expansion of $(1-y)^r$, we obtain:

$$\lim_{r\to 0}\left\{\frac{1}{r}\left[-1+\left(1 + r(-y) + \frac{r(r-1)(-y)^2}{2!} + \frac{r(r-1)(r-2)(-y)^3}{3!} + \frac{r(r-1)(r-2)(r-3)(-y)^4}{4!} + \dots\right)\right]\right\}$$

$$= \lim_{r\to 0}\left\{(-y) + \frac{(r-1)(-y)^2}{2!} + \frac{(r-1)(r-2)(-y)^3}{3!} + \frac{(r-1)(r-2)(r-3)(-y)^4}{4!} + \dots\right\}$$

$$= \lim_{r\to 0}\left\{(-y) - \frac{-(r-1)(-y)^2}{2!} + \frac{(r-1)(r-2)(-y)^3}{3!} - \frac{-(r-1)(r-2)(r-3)(-y)^4}{4!} + \dots\right\}$$

$$= (-y) - \frac{(-y)^2}{2!} + \frac{2!(-y)^3}{3!} - \frac{3!(-y)^4}{4!} + \dots$$

$$= (-y) - \frac{(-y)^2}{2} + \frac{(-y)^3}{3} - \frac{(-y)^4}{4} + \dots$$

which is just the series expansion of $ln(1-y) = ln\,(x)$ (recall that $|-y| < 1$, thus the argument of ln is positive). For the case $x > 1$, one can posit $x = 1/(1-y)$, so that again $|-y| < 1$ and compute $\lim_{r\to 0}\{ln_r(x)\} = \lim_{r\to 0}\left\{\frac{\left(\frac{1}{1-y}\right)^r - 1}{r}\right\} = \lim_{r\to 0}\left\{\frac{1}{r}\left[(1-y)^{-r} - 1\right]\right\}$, thus getting the same result from a similar derivation.

Hence, we will assume $ln_r(x) = ln(x)$ for $r = 0$. Thus defined, the generalized logarithmic function has mathematical meaning for all real values of r, but our main focus in what follows will be on $r \in [-1,1]$. This kind of functions have been employed to generalize the classical (Bolzmann-Shannon) formalism for entropy, with significant applications in information theory, statistical mechanics, and beyond (Havrda and Charvát 1967; Tsallis 1988; Keylock 2005). Our main technical point here is that the whole set of six confirmation measures above can be embedded in the following biparametric continuum:

$$C_{(r,s)}\,(h,e) = ln_s ln_r\,[O\,(h|e) + 1] - ln_s ln_r\,[O(h) + 1]$$

Provided that r, $s \neq 0$, $C_{(r,s)}$ can be further manipulated to yield the following form:

$$C_{(r,s)}\,(h,e) = \frac{1}{s\,(r^s)}\left\{\left[(O\,(h|e) + 1)^r - 1\right]^s - \left[(O(h) + 1)^r - 1\right]^s\right\}$$

We then have:

$$C_{(-1,1)}\,(h,e) = D\,(h,e)$$

$$C_{(0,1)}\,(h,e) = G\,(h,e)$$

$$C_{(1,1)}\,(h,e) = OD\,(h,e)$$

$$C_{(-1,0)}\,(h,e) = R\,(h,e)$$

$$C_{(1,0)}\,(h,e) = OR\,(h,e)$$

$$C_{(-1,-1)}\,(h,e) = C_{(1,-1)}\,(h,e) = ID\,(h,e)$$

A summary representation is given in Fig. 23.1. Some significant implications of this formalism and some interesting issues it raises are addressed in the next section.

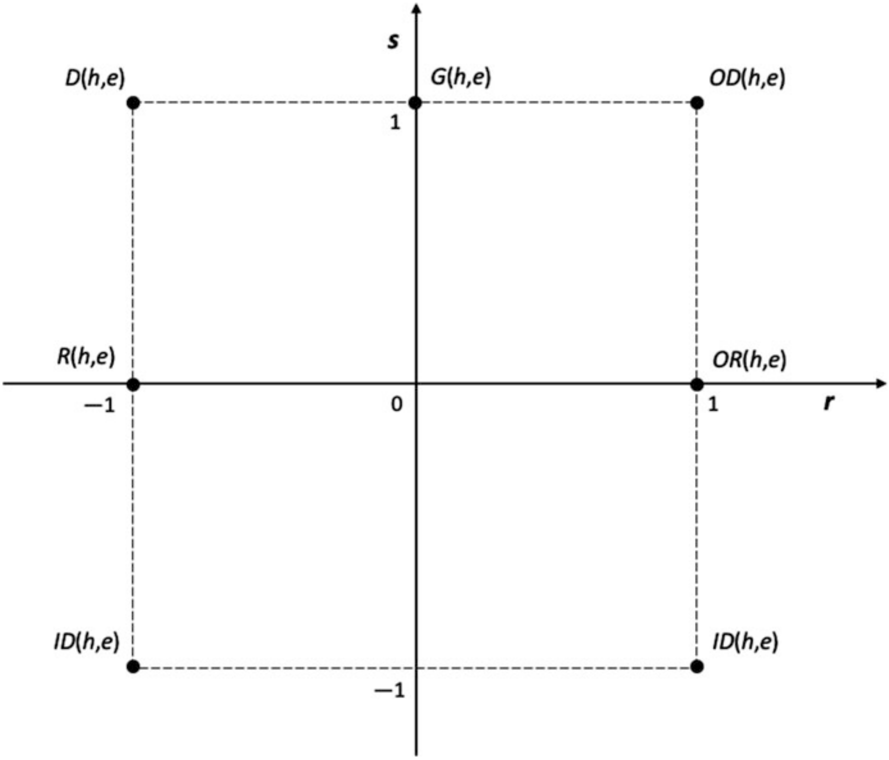

Fig. 23.1 The $C_{(r,s)}$ family of confirmation measures is represented in a Cartesian plane with values of parameter r and of parameter s lying on the x– and y–axis, respectively. Each point in the plane corresponds to a specific confirmation measure. Special cases of interest are highlighted

23.3 Discussion

It is quite easy to verify that the basic features of probabilistic incremental confirmation hold for the whole continuum $C_{(r,s)}$, namely: (i) for any $h,e,f \in \boldsymbol{L_C}$ and any $P \in \boldsymbol{P}$, $C_{(r,s)}(h,e) \gtreqless C_{(r,s)}(h,f)$ if and only if $P(h|e) \gtreqless P(h|f)$, and (ii) for any $h,k \in \boldsymbol{L_C}$ and any $P \in \boldsymbol{P}$, $C_{(r,s)}(h,\top) = C_{(r,s)}(k,\top)$ (0 is the neutrality value). So each instance of $C_{(r,s)}$ is a well-behaved confirmation measure in this fundamental sense.

The role of parameter r in the construction of $C_{(r,s)}$ is perhaps of some interest of its own: *it unifies the probability and the odds formalism*. In fact, for any $a \in \boldsymbol{L_C}$ and any $P \in \boldsymbol{P}$, $ln_r[O(a) + 1] = P(a)$ for $r = -1$ and $ln_r[O(a) + 1] = O(a)$ for $r = 1$.[3] So confirmation measures in $C_{(r,s)}$ relate the prior and posterior

[3] A different way to connect and subsume probabilities and odds was already suggested by Festa (2008). Festa defined a parametric family of "belief functions" $B_\alpha(x) = P(x)/[1 + \alpha P(x)]$ with $\alpha \in [-1,\infty)$, so that $B_{-1}(x) = O(x)$ and $B_0(x) = P(x)$.

values of these generalized credence functions. This explains the apparent puzzle of the $ID(h,e)$ measure, which occurs twice in the parameter space, for both $r = -1$ and $r = 1$. That is because, as already pointed out by Festa and Cevolani (2016), the functional form of $ID(h,e)$ is remarkably invariant across the probability *vs*. odds representation of credences: $1/P(h)-1/P(h|e) = 1/O(h)-1/O(h|e)$. Also of interest, the generalized credence function $ln_r[O(a) + 1]$ has a upper bound (just like probability) for $r < 0$ (the bound being $-1/r$), while it has no upper bound (just like odds) for $r \geq 0$. One worthwhile theoretical idea might be to check whether there exist r-parametrized versions of the probability axioms by which these generalized functions (thus including odds) can be characterized.

A similar issue arises as concerns the following:

$$C_{(r,0)}(h,e) = ln\left\{\frac{ln_r[O(h|e)+1]}{ln_r[O(h)+1]}\right\} = ln\left\{\frac{[O(h|e)+1]^r-1}{[O(h)+1]^r-1}\right\}$$

This is represented by a line along the x axis in Fig. 23.1. Crupi et al. (2013) have provided rather simple axiomatic characterizations of the most prominent special cases of this one-parameter subclass of $C_{(r,s)}$, namely $R(h,e)$ (for $r = -1$) and $OR(h,e)$ (for $r = 1$). Here again, maybe a unified formalism may allow for a more general result and the subsumption of the ones already available.

Another interesting exercise is to fix r instead, and let s vary, as in the following:

$$C_{(-1,s)}(h,e) = ln_s[P(h|e)] - ln_s[P(h)] = \frac{1}{s}\left[P(h|e)^s - P(h)^s\right]$$

This is represented in Fig. 23.1 by the vertical line connecting $ID(h,e)$, $R(h,e)$, and $D(h,e)$. Parameter s determines the specific functional form by which the posterior and prior probabilities, $P(h|e)$ and $P(h)$, are related. The most popular cases—simple algebraic difference and log of the ratio—correspond to $s = 1$ and $s \rightarrow 0$ (in the limit), respectively.

Here, one interesting connection occurs with work on so-called "Matthew effects" in probabilistic confirmation theory. In fact, Festa (2012) and Festa and Cevolani (2016) discussed the Popperian idea that, other things being equal, hypotheses that are initially *less* probable should get a confirmational bonus over more probable ones, to the extent that a lower prior probability indicates greater content and "testability" (also see Roche 2014; Sprenger 2016a). Following Kuipers (2000, p. 25), this may be called an anti-Matthew effect (a Matthew effect being the opposite, i.e., a confirmational advantage for hypotheses with a *higher* prior). Festa (2012) noticed that Matthew and anti-Matthew effects characterize $D(h,e)$ and $ID(h,e)$, respectively, while measure $R(h,e)$ is "Matthew-independent" in his terminology. In our generalized framework, one might thus explore whether, for $r = -1$ (that is, for $C_{(-1,s)}$ as above), $s = 0$ represents a critical threshold to establish the Matthew behavior of a measure, at least for $s \in [-1,1]$. (If so, then perhaps the absolute value of s may serve as a suitable index of how strongly the corresponding measure exhibits Matthew *vs*. anti-Matthew effects, depending on whether s itself is positive *vs*. negative).

23.4 A Straightforward Application to Other Relevance Measures

Following a terminological suggestion by Schippers and Siebel (2015, p. 14), we can label "counterfactual" the following counterpart variants of our six confirmation measures, where prior values $P(h)$ and $O(h)$ are replaced by $P(h|\neg e)$ and $O(h|\neg e)$, respectively:

$$D^* (h, e) = P (h|e) - P (h|\neg e)$$

$$R^* (h, e) = ln \left[\frac{P (h|e)}{P (h|\neg e)} \right]$$

$$G^* (h, e) = ln \left[\frac{P (\neg h|\neg e)}{P (\neg h|e)} \right]$$

$$OD^* (h, e) = O (h|e) - O (h|\neg e)$$

$$ID^* (h, e) = O (\neg h|\neg e) - O (\neg h|e)$$

$$OR^* (h, e) = ln \left[\frac{O (h|e)}{O (h|\neg e)} \right]$$

All of these measures are null for probabilistically independent pairs h,e, and positive *vs*. negative in case h and e are positively *vs*. negatively associated. However, they do *not* generally fulfil the condition that they are higher/equal/lower for h,e as compared to h,f just depending on whether $P(h|e) \gtreqless P(h|f)$ (see Crupi et al. 2007; Climenhaga 2013). So they still are measures of the probabilistic relevance between h and e, but not in the sense of incremental confirmation. Still, most of them are indeed found at various places in the literature. Hájek and Joyce (2008, p. 122), for instance, mention four—$D^*(h,e)$, $R^*(h,e)$, $OD^*(h,e)$, and $OR^*(h,e)$—as candidate measures of "probative value". Moreover, three of these play an important role in contemporary *epidemiology*. For let h be a target occurrence of interest and e a relevant experimental intervention or environmental exposure. Then $D^*(h,e)$ just is the standard measure of the *absolute* change in risk of h due to e and $R^*(h,e)$ an isotone transformation of the *relative* change in risk (see, for example, Barratt et al. 2004). $OR^*(h,e)$, in turn, is simply the log of what is generally known as "the odds ratio" in the epidemiology literature (see A'Court et al. 2012; Cornfield 1951; Milne 2012). Another well-known measure of association, Yule's Q, is also ordinally equivalent to $OR^*(h,e)$ (Garson 2012; Yule 1900). Moreover, according to Fitelson and Hitchcock's (2011) survey, some of these measures have been employed to quantify causal strength (with h now denoting an outcome and e its causal antecedent): Eells (1991) would support $D^*(h,e)$ and Lewis (1986)

$R^*(h,e)$, while both Cheng's (1997) and Good's (1961, 1962) preferred measures are ordinally equivalent to $G^*(h,e)$. Some relevant axiomatic characterizations can be found in Sprenger (2016b). Finally, Schupbach and Sprenger's (2011) favorite measure of explanatory power is ordinally equivalent to $R^*(h,e)$ (also see Crupi and Tentori 2012 and Cohen 2015 for discussion). It is then of potential interest to notice that we can embed all of these measures as special cases of the following, in a way that is strictly parallel to our earlier treatment of incremental confirmation:

$$C^*_{(r,s)}(h,e) = ln_s ln_r [O(h|e) + 1] - ln_s ln_r [O(h|\neg e) + 1]$$

To have a summary illustration, one simply has to refer back to Fig. 23.1 and replace each specific measure by its counterfactual variation, e.g., with $D^*(h,e)$ instead of $D(h,e)$.

References

A'Court, C., R. Stevens, and C. Haneghan. 2012. Against all odds? Improving the understanding of risk reporting. *British Journal of General Practice* 62: e220–e223.

Barratt, A., P.C. Wyer, R. Hatala, T. McGinn, A.L. Dans, S. Keitz, V. Moyer, and G. Guyatt. 2004. Tips for learners of evidence-based medicine, 1: Relative risk reduction, absolute risk reduction, and number needed to treat. *Canadian Medical Association Journal* 171: 353–358.

Brössel, P. 2013. The problem of measure sensitivity redux. *Philosophy of Science* 80: 378–397.

Carnap, R. 1950. *Logical foundations of probability*. Chicago: University of Chicago Press.

Chandler, J. 2013. Contrastive confirmation: Some competing accounts. *Synthese* 190: 129–138.

Cheng, P. 1997. From covariation to causation: A causal power theory. *Psychological Review* 104: 367–405.

Climenhaga, N. 2013. A problem for the alternative difference measure of confirmation. *Philosophical Studies* 164: 643–651.

Cohen, M.P. 2015. On three measures of explanatory power with axiomatic representations. *British Journal for the Philosophy of Science*. doi:10.1093/bjps/axv017.

Cornfield, J. 1951. A method for estimating comparative rates from clinical data. Applications to cancer of the lung, breast, and cervix. *Journal of the National Cancer Institute* 11: 1269–1275.

Crupi, V. 2015. Inductive logic. *Journal of Philosophical Logic* 44: 641–650.

Crupi, V., and K. Tentori. 2012. A second look at the logic of explanatory power (with two novel representation theorems). *Philosophy of Science* 79: 365–385.

Crupi, V., N. Chater, and K. Tentori. 2013. New axioms for probability and likelihood ratio measures. *British Journal for the Philosophy of Science* 64: 189–204.

———. 2013. Confirmation as partial entailment: A representation theorem in inductive logic. *Journal of Applied Logic* 11: 364–372. [Erratum in *Journal of Applied Logic* 12: 230–231].

———. 2014. State of the field: Measuring information and confirmation. *Studies in the History an Philosophy of Science A* 47: 81–90.

Crupi, V., K. Tentori, and M. Gonzalez. 2007. On Bayesian measures of evidential support: Theoretical and empirical issues. *Philosophy of Science* 74: 229–252.

Crupi, V., R. Festa, and C. Buttasi. 2010. Towards a grammar of Bayesian confirmation. In *Epistemology and methodology of science*, ed. M. Suárez, M. Dorato, and M. Rédei, 73–93. Dordrecht: Springer.

Eells, E. 1991. *Probabilistic causality*. Cambridge: Cambridge University Press.

Festa, R. 1999. Bayesian confirmation. In *Experience, reality, and scientific explanation*, ed. M. Galavotti and A. Pagnini, 55–87. Dordrecht: Kluwer.

———. 2008. On the Matthew effect and other properties of Bayesian confirmation [talk]. *Workshop on probability, confirmation, and fallacies*, University of Leuven, April 6, 2008.

———. 2012. "For unto every one that hath shall be given": Matthew properties for incremental confirmation. *Synthese* 184: 89–100.

Festa, R., and G. Cevolani 2016. Unfolding the grammar of Bayesian confirmation: Likelihood and anti-likelihood principles. *Philosophy of Science,* forthcoming.

Fitelson, B. 2001. A Bayesian account of independent evidence with applications. *Philosophy of Science* 68: S123–S140.

———. 2007. Likelihoodism, Bayesianism, and relational confirmation. *Synthese* 156: 473–489.

Fitelson, B., and C. Hitchcock. 2011. Probabilistic measures of causal strength. In *Causality in the sciences*, ed. P. McKay Illari, F. Russo, and J. Williamson, 600–627. Oxford: Oxford University Press.

Gaifman, H. 1979. Subjective probability, natural predicates, and Hempel's ravens. *Erkenntnis* 14: 105–147.

Garson, G.D. 2012. *Measures of association*. Asheboro: Statistical Associates Publishers.

Glass, D.H. 2013. Confirmation measures of association rule interestingness. *Knowledge-Based Systems* 44: 65–77.

Good, I.J. 1950. *Probability and the weighing of evidence*. London: Griffin.

———. 1961. A causal calculus I. *British Journal for the Philosophy of Science* 11: 305–318.

———. 1962. A causal calculus II. *British Journal for the Philosophy of Science* 12: 43–51.

Hájek, A., and J. Joyce. 2008. Confirmation. In *Routledge companion to the philosophy of science*, ed. S. Psillos and M. Curd, 115–129. New York: Routledge.

Havrda, J., and F. Charvát. 1967. Quantification method of classification processes: Concept of structural *a*-entropy. *Kybernetica* 3: 30–35.

Heckerman, D. 1988. An axiomatic framework for belief updates. In *Uncertainty in artificial intelligence 2*, ed. J.F. Lemmer and L.N. Kanal, 11–22. Amsterdam: North-Holland.

Howson, C. 2000. *Hume's problem: Induction and the justification of belief*. Oxford: Oxford University Press.

Iranzo, V., and I. Martínez de Lejarza. 2012. On ratio measures of confirmation. *Journal for General Philosophy of Science* 44: 193–200.

Joyce, J. 2004. Bayes's theorem. In ed. E.N. Zalta, *The Stanford encyclopedia of philosophy* (Summer 2004 Edition). URL:http://plato.stanford.edu/archives/sum2004/entries/bayes-theorem/

Keylock, J.C. 2005. Simpson diversity and the Shannon-Wiener index as special cases of a generalized entropy. *Oikos* 109: 203–207.

Kuipers, T. 2000. *From instrumentalism to constructive realism*. Dordrecht: Reidel.

Lewis, D. 1986. Postscripts to 'causation'. In *Philosophical papers*, vol. II, 173–213. Oxford: Oxford University Press.

Milne, P. 1996. Log[$P(h|eb)/P(h|b)$] is the one true measure of confirmation. *Philosophy of Science* 63: 21–26.

———. 2012. On measures of confirmation. Manuscript.

Park, I. 2014. Confirmation measures and collaborative belief updating. *Synthese* 191: 3955–3975.

Pruss, A.R. 2013. Probability, regularity, and cardinality. *Philosophy of Science* 80: 231–240.

Rips, L. 2001. Two kinds of reasoning. *Psychological Science* 12: 129–134.

Roche, W. 2014. A note on confirmation and Matthew properties. *Logic & Philosophy of Science* 12: 91–101.

Roche, W., and T. Shogenji. 2014. Dwindling confirmation. *Philosophy of Science* 81: 114–137.

Royall, R. 1997. *Statistical evidence: A likelihood paradigm*. London: Chapman & Hall.

Schippers, M., and M. Siebel. 2015. Inconsistency as a touchstone for coherence measures. *Theoria* 30: 11–41.

Schupbach, J.N., and J. Sprenger. 2011. The logic of explanatory power. *Philosophy of Science* 78: 105–127.

Sober, E. 1990. Contrastive empiricism. In *Minnesota studies in the philosophy of science: Scientific theories*, vol. 14, ed. C.W. Savage, 392–412. Minneapolis: University of Minnesota Press.

Sprenger, J. 2016a. Two impossibility results for measures of corroboration. *British Journal for the Philosophy of Science*, forthcoming.

———. 2016b. *Foundations for a probabilistic theory of causal strength*. See: http://philsci-archive.pitt.edu/11927/1/GradedCausation-v2.pdf.

Tsallis, C. 1988. Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics* 52: 479–487.

Yule, G.U. 1900. On the association of attributes in statistics, with illustrations from the material from the childhood society. *Philosophical Transactions A* 194: 257–319.