CHAPTER 30

CONFIRMATION THEORY

VINCENZO CRUPI AND KATYA TENTORI

In philosophy of science, formal epistemology, and related areas, *confirmation* has become a key technical term. Broadly speaking, confirmation has to do with how evidence affects the credibility of hypotheses, an issue that is crucial to human reasoning in a variety of domains, from scientific inquiry to medical diagnosis, legal argumentation, and beyond. In what follows, we will address *probabilistic* theories of confirmation. The case for tackling confirmation in a probabilistic framework is easily put. The connection between evidence and hypothesis is typically fraught with uncertainty, and probability is widely recognized as the formal representation of uncertainty that is best understood and motivated.[1] We will thus frame our discussion by positing a set $\boldsymbol{P}$ of probability functions representing possible states of belief concerning a domain described in a (finite) propositional language L. We will also denote as L_c the set of contingent formulae in L (namely, those expressing neither logical truths nor logical falsehoods), and we will have hypothesis h and evidence e belonging to L_c. Finally, $\boldsymbol{P}$ will be assumed to include all *regular* probability functions that can be defined over L (i.e., such that, for any $\alpha \in L_c$ and any $P \in \boldsymbol{P}$, $0 < P(\alpha) < 1$).[2]

[1] Although well-established, probabilistic confirmation theory has not always been popular, nor has it remained unchallenged even in recent times. For prominent critical voices, see Kelly and Glymour (2004) and Norton (2010). As regards earlier influential and non-probabilistic accounts of confirmation, one should mention at least Popper's (1959) notion of "corroboration" through bold successful predictions and Hempel's (1943) analysis of confirmation by instances. There also exist cases which tend to defy the distinction between advocates and critics of probabilistic confirmation theory: Isaac Levi's work is a major example (e.g., Levi 2010). Finally, there are authors who rely on probability to account for evidential reasoning, but not as a representation of belief under uncertainty (as is the case throughout this chapter). This applies, for instance, to Royall's (1997) likelihoodism, as well as to Mayo's (1996) error-theoretic approach. Also see Crupi (2015) for a more extensive discussion.

[2] Regularity can be motivated as a way to represent credences that are non-dogmatic (see Howson 2000: p. 70). It is a very convenient assumption, but not an entirely innocent one. Festa (1999) and Kuipers (2000) discuss some limiting cases that are left aside here owing to this constraint.

30.1 Qualitative Confirmation Absolute vs. Incremental

A *qualitative* account of confirmation amounts to spelling out the conditions on which evidence e does or does not confirm hypothesis h. On the qualitative level of analysis, a clear distinction must be drawn between so-called *absolute* and *incremental* confirmation (see, e.g., Hájek and Joyce 2008). Adapting a useful piece of formalism (originally due to Gabbay 1982, and now standard in non-monotonic logics), we will employ "$|\!\sim^{A}_{P}$" for "confirms in the absolute sense (relative to P)" and "$|\!\sim^{I}_{P}$" for "confirms in the incremental sense (relative to P)".

(Abs) *Absolute confirmation*
For any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, $e \; |\!\sim^{A}_{P} \; h$ if and only if $P(h|e) > r$ (with $½ \leq r$).

(Inc) *Incremental confirmation*
For any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, $e \, |\!\sim^{I}_{P} \, h$ if and only if $P(h|e) > P(h)$.

Absolute confirmation, as defined above, concerns whether or not the probability of h given e is *high enough* relative to a threshold value r. (This value must be separately specified and can set up a more or less demanding criterion.) On the other hand, incremental confirmation concerns whether or not the probability of h is *increased* when e is acquired as evidence. Before presenting and discussing critical divergences between them, let us point out that absolute and incremental confirmation share some rather basic properties. There follows a list of four, all of which are implied by each of both (Abs) and (Inc). Since they hold for absolute and incremental confirmation alike, superscripts A and I are removed.[3] (Also, notation is as expected, in that we write $\{\alpha, \beta\} \, |\!\sim_P \, \gamma$ if and only if $(\alpha \wedge \beta) \, |\!\sim_P \, \gamma$).

(EC) *Entailment condition*
For any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, if $e \models h$, then $e \; |\!\sim_P \; h$.

(NM) *Non-monotonicity*
For any $h,e \in L_c$ and any $P \in \boldsymbol{P}$ such that $e \, |\!\sim_P \, h$ and $e \not\models h$, there exists $x \in L_c$ such that $\{e,x\} \not|\!\sim_P \, h$.[4]

(Cases) *Proof by cases*
For any $h,e,x \in L_c$ and any $P \in \boldsymbol{P}$, if $\{e,x\} \; |\!\sim_P \; h$ and $\{e,\neg x\} \, |\!\sim_P \, h$, then $e \; |\!\sim_P \; h$.

(CComp) *Confirmation complementarity* (*qualitative*)

For any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, if $e \; |\!\sim_P \; h$ then $e \not|\!\sim_P \neg h$.

[3] (EC) has been standard ever since Hempel (1945: p. 103) and it is analoguous to so-called *superclassicality* in logical parlance (see, e.g., Antonelli 2012). (NM) is inspired by Fitelson and Hawthorne (2010: p. 209). See Malinowski (2005) and Kuipers (2007) for earlier appearances of (Cases), and Crupi, Festa, and Buttasi (2010: p. 85) for remarks and terminology relevant to (CComp).

[4] The easiest way to prove (NM) is to just posit $x = \neg h$.

All four of the above conditions seem compelling upon reflection. First, relations of plain deductive entailment are instances of confirmation, as stated by (EC). Here, of course, hypothesis h is *conclusively* established in light of the evidence e, so that these are "ideal" and special instances, as it were. Indeed, confirmation is otherwise a form of non-monotonic reasoning in the sense of (NM), thus *non*-conclusive and defeasible. This is as it should be, motivated by the consideration that a hypothesis (say, Newtonian physics) can receive spectacular confirmation and nevertheless be overthrown in light of subsequent further evidence. However, as (Cases) implies, if confirmation of h happens not to be defeated by conjoining e to either of the statements x or $\neg x$, then e confirms h regardless. Finally, the claim that some evidence e confirms both hypothesis h and its negation $\neg h$ would be unintelligible, so that (CComp) also seems an obvious requirement.

Despite these preliminary remarks, it is important to realize that absolute and incremental confirmation convey very different concepts. Indeed, the distinction between the two – "extremely fundamental" and yet "sometimes unnoticed", as Salmon (1969: pp. 48–9) put it – has proved recurrently necessary for theoretical clarity (see, e.g., Crupi, Fitelson, and Tentori 2008). The following distinctive properties of $\mid\!\sim^{A}_{P}$ (both reaching back to Hempel 1945: pp. 103 ff.) will help us develop this point more thoroughly.

(SC) *Special consequence condition*
For any $h_1,h_2,e \in L_c$ and any $P \in \boldsymbol{P}$, if $h_1 \models h_2$ and $e \mid\!\sim^{A}_{P} h_1$, then $e \mid\!\sim^{A}_{P} h_2$.

(CC) *Consistency condition*
For any $h_1,h_2,e \in L_c$ and any $P \in \boldsymbol{P}$, if $\models \neg(h_1 \wedge h_2)$ and $e \mid\!\sim^{A}_{P} h_1$, then $e \not\mid\!\sim^{A}_{P} h_2$.

In most contexts, "confirming evidence" is taken to be evidence which "makes a difference," to some extent at least, in favor of the hypothesis of interest. Bearing this in mind, it is then easy to show that (SC) is too inclusive, while (CC) is too restrictive. As a consequence, although a formally unobjectionable and historically influential notion, $\mid\!\sim^{A}_{P}$ does not seem to characterize confirmation very effectively. Let us discuss this line of argument in more detail.

As to the assessment of (SC), a simple numerical example will best serve our purposes. Suppose that a card is drawn from a well shuffled standard deck. Let h_1 be "the card drawn is a red non-face card" and let h_2 be "the card drawn is a non-face card," so that h_1 implies h_2. If the evidence e is provided that the card drawn is actually red, then it seems natural to observe that h_1, but not h_2, receives support, and is thereby confirmed. By (SC), on the contrary, confirmation must extend to any consequence of h_1, including h_2, so that here $\mid\!\sim^{A}_{P}$ lets in too much. This illustrates a much more general concern. In fact, for *any* pair of unrelated (independent) statements x,y such that the latter is likely enough in its own terms, we will have $x \mid\!\sim^{A}_{P} (x \wedge y)$ and thus, by (SC), $x \mid\!\sim^{A}_{P} y$ too.

Let us now turn to (CC). This states that evidence e can never confirm incompatible hypotheses. But consider, by way of illustration, a clinical case of an infectious disease of unknown origin, and suppose that e is the failure of antibiotic treatment. There seems to be nothing wrong in saying that, by discrediting bacteria as possible causes, the evidence confirms (viz. provides support for) any of a number of alternative viral diagnoses. This would not be allowed by (CC), however; so that here $\mid\!\sim^{A}_{P}$ lets in too little.

In contrast to the foregoing, the following distinctive principles of *incremental* confirmation show that this notion matches widespread patterns of reasoning about evidence and hypotheses.

(CE) *Converse entailment condition*
For any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, if $h \models e$, then $e \mid\!\sim^I_P h$.

(EComp) *Complementary evidence*
For any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, if $e \mid\!\sim^I_P h$ then $\neg e \not\mid\!\sim^I_P h$.

Condition (CE) (the label once again comes from Hempel 1945: p. 104) naturally conveys the statement that hypotheses are confirmed by their consequences that are borne out by observation, this being a paramount precept of scientific methodology. Notably, this elementary principle would not be licenced by the relation of absolute confirmation $\mid\!\sim^A_P$.

Condition (EComp) is no less relevant. Consider the following example. A father is suspected of abusing his child. Suppose that the child does indeed claim that s/he has been abused (label this piece of evidence e). A forensic psychiatrist, when consulted, declares that this confirms guilt (h). Alternatively, suppose that the child is asked and does *not* report having been abused ($\neg e$). As pointed out by Dawes (2001), it may well happen that a forensic psychiatrist will nonetheless interpret *this* as evidence confirming guilt (suggesting that violence has prompted the child's denial). One might want to argue that this kind of "heads I win, tails you lose" judgment would be inconsistent, and thus untenable on a purely logical basis. Whoever concurs with this line of argument (as Dawes 2001 himself did) must be presupposing the incremental, not the absolute, notion of confirmation. The latter would not do, in fact, for it is easy to show that $e \mid\!\sim^A_P h$ and $\neg e \mid\!\sim^A_P h$ can obtain concurrently. Condition (EComp), on the other hand, prescribes that *only one* of the contradictory statements e and $\neg e$ can (incrementally) confirm a hypothesis h.

Remarks such as the foregoing have induced some contemporary theorists to dismiss the very notion of absolute confirmation, concluding that "if you had $P(h|e)$ close to unity [i.e., $e \mid\!\sim^A_P h$, in our current notation], but less than $P(h)$ [i.e., $e \not\mid\!\sim^I_P h$], you *ought not* to say that h was confirmed by e" (Good 1968: p. 134; see also Salmon 1975: p. 13). In the remainder of this chapter, we will comply with this suggestion and focus on confirmation in the incremental sense throughout.

30.2 The Axiomatics of Quantitative Confirmation

Assessments of the *amount* of support that a piece of evidence brings to a hypothesis are commonly required in scientific reasoning, as well as in other domains, if only in the form of comparative judgments such as "hypothesis h is more strongly confirmed by e_1 than by e_2" or "e confirms h_1 to a greater extent than h_2." A purely qualitative theory of confirmation is not up to the challenge of providing a foundation for judgments of this kind. However, a probabilistic approach does allow for a proper quantitative treatment, i.e., the definition of a *measure* $C_P(h,e)$: $\{L_c \times L_c \times \boldsymbol{P}\} \rightarrow \Re$ of the degree of confirmation that h receives from e relative to P. (Indeed, as we shall see shortly, a wealth of such measures can be proposed.)

As we want a confirmation measure $C_P(h,e)$ to have relevant probabilities as its building blocks, the following background assumption is in order:

(F) *Formality*
There exists a function g such that, for any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, $C_P(h,e) = g[P(h \wedge e), P(h), P(e)]$.

Note that the probability distribution over the algebra generated by h and e is entirely determined by $P(h \wedge e)$, $P(h)$, and $P(e)$. Hence (F) simply states that $C_P(h,e)$ depends on that distribution, and nothing else. This is a widespread assumption in discussions of confirmation in a probabilistic framework, although it is often tacit or spelled out in slightly different ways. (The label *formality* is taken from Tentori, Crupi, and Osherson 2007, 2010).

Another preliminary constraint is sometimes defined along the following lines:

(D) *Discrimination*
There exists $t \in \Re$ such that, for any $h,e \in L_c$ and any $P \in \boldsymbol{P}$:

(i) $C_P(h,e) > t$ if and only if $e \mathrel{|\!\!\sim^I_P} h$;
(ii) $C_P(h,e) < t$ if and only if $e \mathrel{|\!\!\sim^I_P} \neg h$;
(iii) $C_P(h,e) = t$ if and only if $e \mathrel{|\!\!\not\sim^I_P}$ and $e \mathrel{|\!\!\not\sim^I_P} \neg h$.

Principle (D) states that a fixed figure t acts as a threshold separating cases in which e confirms h (thus *dis*confirming $\neg h$, as we will say hereafter) from cases in which e confirms $\neg h$ (thus *dis*confirming h). The value t itself indicates *neutrality* (of evidence e relative to h vs. $\neg h$) and is set as a matter of convenience, usual choices being 0 or 1. Condition (D) suffices to guarantee that the foregoing properties of $|\!\!\sim^I_P$– as conveyed by (EC), (NM), (Cases), (CComp), (CE), and (EComp) above – are all retained under $C_P(h,e)$, thus fulfilling a natural constraint of coherence between the purely qualitative notion and its quantitative refinement. (D) is a rather mild requirement, however, for there exist functions of all sorts that satisfy it. Historically, the outlook of theorists for the representation of $C_P(h,e)$ has been much more selective. The most popular candidates have in fact amounted to the following:[5]

Probability difference: $P(h|e) - P(h)$
Probability ratio: $P(h|e)/P(h)$
Likelihood ratio: $P(e|h)/P(e|\neg h)$

Although they are all consistent with (D), the above quantities differ substantially in that they are *not ordinally equivalent*. Two confirmation measures are said to be ordinally equivalent if they *always rank* evidence–hypothesis pairs in the same way. More formally, $C_P(h,e)$ and $C^*_P(h,e)$ are ordinally equivalent if and only if, for any $h_1,h_2,e_1,e_2 \in L_c$ and any $P \in \boldsymbol{P}$, $C_P(h_1,e_1) \gtreqless C_P(h_2,e_2)$ if and only if $C^*_P(h_1,e_1) \gtreqless C^*_P(h_2,e_2)$. Isotone transformations of a given quantity yield measures whose detailed quantitative behavior (including range

[5] The probability difference has been first defined by Carnap (1950/1962: p. 361), the probability ratio by Keynes (1921: pp. 165 ff.), and the likelihood ratio by Alan Turing (as reported in Good 1950: pp. 62–3).

and neutrality value) may vary widely, but such that rank-order is strictly preserved. For instance, the measures in the following list are all ordinally equivalent variants based on the probability ratio:[6]

$r_0(h,e) = P(h\|e)/P(h)$	range: $[0,+\infty)$	neutrality value: 1
$r_1(h,e) = \frac{P(h\|e)-P(h)}{P(h)}$	range: $[-1,+\infty)$	neutrality value: 0
$r_2(h,e) = \ln[P(h\|e)/P(h)]$	range: $[-\infty,+\infty)$	neutrality value: 0
$r_3(h,e) = \frac{P(h\|e)-P(h)}{P(h\|e)+P(h)}$	range: $[-1,1)$	neutrality value: 0
$r_4(h,e) = \frac{P(h\|e)}{P(h\|e)+P(h)}$	range: $[0,1)$	neutrality value: ½

The ordinal divergence among alternative confirmation measures is arguably of greater theoretical significance than purely quantitative differences, because the former, unlike the latter, implies opposite comparative judgments for some evidence-hypothesis pairs. Indeed, in what follows we will deal *only* with the ordinal level in the assessment of confirmation. We will thus invariably address properties that apply to $C_P(h,e)$ if and only if they also apply to any ordinally equivalent $C^*_P(h,e)$ and treat classes of ordinal equivalence as our unit of analysis. Accordingly, we will simply say that $C_P(h,e)$ is a *probability difference measure* if and only if there exists a strictly increasing function f such that $C_P(h,e) = f[P(h|e) - P(h)]$, and the same for the probability and likelihood ratio.

An effective tool to gain theoretical insight concerning (ordinally) different measures of confirmation is to provide an exhaustive set of axioms for each of them. It turns out that four fundamental statements, along with the basic requirement of formality (F), suffice to distinguish neatly among the traditional options considered above.[7]

(C1) *Final probability*
For any $h,e_1,e_2 \in L_c$ and any $P \in \mathbf{P}$, $C_P(h,e_1) \gtreqless C_P(h,e_2)$ if and only if $P(h|e_1) \gtreqless P(h|e_2)$.

(C2) *Disjunction of alternative hypotheses*
For any $h_1,h_2,e \in L_c$ and any $P \in \mathbf{P}$, if $P(h_1 \wedge h_2) = 0$, then $C_P(h_1 \vee h_2,e) \gtreqless C_P(h_1,e)$ if and only if $P(h_2|e) \gtreqless P(h_2)$.

(C3) *Law of likelihood*
For any $h_1,h_2,e \in L_c$ and any $P \in \mathbf{P}$, $C_P(h_1,e) \gtreqless C_P(h_2,e)$ if and only if $P(e|h_1) \gtreqless P(e|h_2)$.

(C4) *Modularity (for conditionally independent data)*
For any $h,e_1,e_2 \in L_c$ and any $P \in \mathbf{P}$, if $P(e_1|\pm h \wedge e_2) = P(e_1|\pm h)$, then $C_P(h,e_1|e_2) = C_P(h,e_1)$.[8]

(C1) states that, for any hypothesis h, final probability and confirmation always move in the same direction in the light of data, e. This seems a very compelling principle, and it is

[6] Obviously, $r_1(h,e) = r_0(h,e) - 1$ and $r_2(h,e) = \ln[r_0(h,e)]$. Moreover, $r_3(h,e) = [r_0(h,e) - 1]/[r_0(h,e) + 1]$ and $r_4(h,e) = r_0(h,e)/[r_0(h,e) + 1]$.

[7] For recent occurrences of (C1), see Fitelson (2006: p. 506) and Hájek and Joyce (2008: p. 122). (C3) is endorsed by both Edwards (1972: pp. 30–1) and Milne (1996). The label "law of likelihood" goes back to Hacking (1965), while that for (C4) is freely adapted from Heckerman (1988: pp. 18–19).

[8] The notion of conditional confirmation denoted by $C_P(h,e_1|e_2)$ implies that all relevant values from P are conditionalized on e_2. The expression "$\pm h$" is meant to cover two cases, i.e., both the statement and the negation of h. In some contexts, condition $P(e_1|\pm h \wedge e_2) = P(e_1|\pm h)$ is also referred to as *screening off* (of e_1 and e_2 by h).

in fact the only condition among the foregoing that has remained virtually unchallenged.[9] On the other hand, the choice among the competing measures listed essentially depends on the acceptance of either (C2), (C3), or (C4), as shown by the result below. (Note that one does not need to assume the fundamental *Discrimination* condition (D) separately, for it follows in any of the three clauses of the theorem and thus becomes formally redundant.)

Theorem 1

(i) *(F), (C1) and (C2) if and only if $C_P(h,e)$ is a probability difference measure.*
(ii) *(F), (C1) and (C3) if and only if $C_P(h,e)$ is a probability ratio measure.*
(iii) *(F), (C1) and (C4) if and only if $C_P(h,e)$ is a likelihood ratio measure.*

A proof of clause (i) of Theorem 1 is given in the Appendix, while clauses (ii) and (iii) are proven in Crupi, Chater, and Tentori (2013).[10]

The plurality of probabilistic measures of confirmation has prompted some scholars to be skeptical or dismissive of the prospects for a quantitative theory of confirmation (see, e.g., Howson 2000: pp. 184–5, and Kyburg and Teng 2001: pp. 98 ff.). However, quantitative probabilistic analyses have proved crucial for handling a number of puzzles and issues that plagued more qualitative approaches, including the so-called "irrelevant conjunction" problem, Hempel's paradox of the ravens, Goodman's "new riddle of induction", the variety of evidence, and the Duhem-Quine thesis (see Earman 1992: pp. 63–117 for a now classic discussion, and Crupi 2015 for a more recent survey). And in fact, various arguments in the philosophy of science have been shown to depend critically (and sometimes unwittingly) on the choice of one confirmation measure (or some of them) rather than others (Festa 1999; Fitelson 1999; Brössel 2013). Relying on the appeal of distinctive features, some authors have insisted on "one true measure" of confirmation (see Good 1984; Milne 1996; but also see Milne 2012), while others have seen different measures as possibly capturing "distinct,

[9] Precisely for this reason, we forgo detailed treatment of candidate measures departing from (C1). To be noted, however, that among these Carnap's (1950/1962: p. 360) measure $P(h \wedge e) - P(h)P(e)$ implies (C2), Mortimer's (1988: Section 11.1) measure $P(e|h) - P(e)$ implies (C3), and Nozick's (1981: p. 252) measure $P(e|h) - P(e|\neg h)$ implies (C4). Indeed, the corresponding classes of ordinally equivalent measures can be axiomatized much as in our Theorem 1, provided that (C1) is replaced with the following (proofs omitted):

(C1*) Disjunction of alternative data
For any $h, e_1, e_2 \gtreqless \in L_c$ and any $P \in \mathbf{P}$, if $P(e_1 \wedge e_2) = 0$, then $C_P(h|e_1 \vee e_2) \gtreqless C_P(h,e_1)$ if and only if $P(h|e_2) \gtreqless P(h)$.

[10] In line with Carnap's (1950/1962) classic work, the standard quantitative counterpart of *absolute* qualitative confirmation (namely, relation $\mid\!\sim^A_P$ from the preceding section) is $P(h|e)$ itself. One can thus wonder whether also this notion is amenable to a similar axiomatic treatment. To see that this is in fact the case, consider the following condition (already put forward by Törnebohm 1966: p. 81):

(A) For any $h,e \in L_c$ and any $P \in \mathbf{P}$, $C_P(h,e) = C_P(h \wedge e, e)$.

It can then be shown that (F), (C1) and (A) hold if and only if there exists a strictly increasing function f such that $C_P(h,e) = f[P(h|e)]$ (see Schippers 2016). Condition (A) seems indeed plausible *if* (but *only if*, in our view) the overall credibility of the hypothesis on the evidence is at issue (as contrasted with the *impact* on the credibility of the hypothesis yielded by the evidence). Accordingly, (A) is inconsistent with (D) above.

complementary notions of evidential support" (Hájek and Joyce 2008: p. 123; also see Huber 2008).

We find the latter approach sensible, but suggest that pluralism be supplemented and tempered by critical scrutiny (see Steel 2007 for a similar position). The axiomatic characterization of competing measures seems particularly useful for this purpose. By way of illustration, once again consider a draw from a standard well shuffled deck and posit h_1 = "the card drawn is 7♠", h_2 = "the card drawn is red", and e = "the card drawn is a face". Note that here $P(h_2|e) = P(h_2)$, while the antecedent of (C2) is satisfied, so according to this principle $C_P(h_1,e) = C_P(h_1 \vee h_2,e)$, even if h_1 is conclusively disconfirmed (i.e., plainly refuted) by e, while $h_1 \vee h_2$ is not (for any red face card would still make it true notwithstanding e). This implication might well seem disturbing, thus speaking against difference measures of confirmation. For, as pointed out by Zalabardo (2009: p. 631), a "choice between […] accounts of confirmation should be dictated by the plausibility of the orderings they generate". For a (critical) discussion of (C3) and a (supportive) examination of (C4), both carried out in a similar vein, we refer the reader to Fitelson (2007 and 2001, respectively).

30.3 Confirmation as Partial Entailment Relative Distance Measures

It has often been maintained that confirmation theory should yield an *inductive logic* that is analogous to classical deductive logic in some suitable sense. This view has been pursued in a number of variants, mostly depending, as Hawthorne (2014) has observed, on "precisely how the deductive model is emulated". According to an old and illustrious idea, the deductive model should be paralleled by a generalized, quantitative theory of *partial entailment*. The following revealing passage, again from Hawthorne (2014), attests to the enduring influence of this notion, albeit from a pessimistic perspective:

> A collection of premise sentences *logically entails* a conclusion sentence just when the negation of the conclusion is *logically inconsistent* with those premises. An inductive logic must, it seems, deviate from this paradigm […]. Although the notion of *inductive support* is analogous to the deductive notion of *logical entailment*, and is arguably an extension of it, there seems to be no inductive logic extension of the notion of *logical inconsistency* – at least none that is interdefinable with *inductive support* in the way that *logical inconsistency* is interdefinable with *logical entailment*.

The central point of the present section amounts to showing that this resignation is overly hasty. As we shall see, it is perfectly possible to have a sound extension of the notion of logical inconsistency that is indeed interdefinable with inductive support in essentially the same way that logical inconsistency is interdefinable with logical entailment. So much so, in fact, that one can safely and fruitfully embed into axioms those very properties which inductive logic would inevitably lack, according to Hawthorne's line of argument. To show this, we first need to introduce a new class of confirmation measures. Consider the following function:

$$z(h,e) = \begin{cases} \frac{P(h|e)-P(h)}{1-P(h)} & \text{if } P(h|e) \geq P(h) \\ \frac{P(h|e)-P(h)}{P(h)} & \text{if } P(h|e) < P(h) \end{cases}$$

Despite its twofold algebraic form, measure $z(h,e)$ conveys a unifying core intuition.[11] More precisely, in case of (positive) confirmation, $z(h,e)$ measures how far upward the posterior $P(h|e)$ has gone in covering the distance between the prior $P(h)$ and 1; that is, it expresses the relative reduction of the initial distance from certainty of h being true as yielded by e. Similarly, in the case of *dis*confirmation, $z(h,e)$ measures how far downward the posterior $P(h|e)$ has gone in covering the distance between the prior $P(h)$ and 0; that is, it reflects the relative reduction of the initial distance from certainty of h being false as yielded by e. So $z(h,e)$ measures the extent to which the initial probability distance from certainty concerning the truth/falsehood of h is reduced by the confirming/disconfirming statement e. Or, put otherwise, how much of such distance is "covered" by the upward/downward jump from $P(h)$ to $P(h|e)$. $z(h,e)$ is thus a measure of the relative reduction of the distance from certainty that a hypothesis of interest is true or false – or, in short (and abusing language a little), a *relative distance* measure.[12] Accordingly, we will say that $C_P(h,e)$ is a *relative distance measure* if and only if there exists a strictly increasing function f such that $C_P(h,e) = f[z(h,e)]$.[13]

Relying again on an axiomatic approach, we now show how relative distance measures escape Hawthorne's (2014) pessimistic conclusion. Drawing from the quote above, we will first assume $C_P(h,e)$ to exhibit a commutative behavior whenever h and e are inductively at odds (i.e., negatively correlated), thus paralleling the symmetric nature of logical inconsistency, as follows:

(C5) *Partial inconsistency*
For any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, if $P(h \wedge e) \leq P(h)P(e)$, then $C_P(h,e) = C_P(e,h)$.

An unrestricted form of commutativity has appeared as a basic and sound requirement in probabilistic analyses of degrees of "coherence" (and lack thereof).[14] In (C5), however, commutativity is not meant to extend to the quantification of *positive* confirmation or support, because logical entailment (unlike refutation) is not symmetric; nor is it coextensive with logical equivalence (or mere logical consistency, for that matter) in the way that refutation is coextensive with inconsistency (Eells and Fitelson 2002 and Crupi, Tentori, and Gonzalez 2007 discuss this point further).

As for the interdefinability of logical entailment of h from e and inconsistency of e with $\neg h$, it naturally generalizes to an inverse (ordinal) correlation between positive confirmation and partial inconsistency with regard to complementary hypotheses, as follows:

(C6) *Confirmation complementarity* (*ordinal*)
For any $h_1,h_2,e \in L_c$ and any $P \in \boldsymbol{P}$, $C_P(h_1,e) \gtreqless C_P(h_2,e)$ if and only if $C_P(\neg h_1,e) \lesseqgtr C_P(\neg h_2,e)$.

[11] An alternative, more compact rendition is the following:

$$z(h,e) = \frac{\min[P(h|e),P(h)]}{P(h)} - \frac{\min[P(\neg h|e),P(\neg h)]}{P(\neg h)}$$

In this form, $z(h,e)$ is structurally similar to Mura's (2006, 2008) measure of "partial entailment". Mura's measure and $z(h,e)$, however, are demonstrably non-equivalent in ordinal terms (see Crupi and Tentori 2014).

[12] This label was first adopted by Huber (2007).

[13] For more extensive discussion and some additional relevant references, see Crupi, Tentori, and Gonzalez (2007), Crupi, Festa, and Buttasi (2010), and Crupi and Tentori (2010, 2013).

[14] See Shogenji's (1999) seminal work. For updated and informed discussions of subsequent developments, see Schupbach (2011) and Schippers (2014).

Indeed, (C6) can be viewed as a fairly faithful formal rendition of Keynes' (1921: p. 80) remark that "an argument is always as near to proving or disproving a proposition, as it is to disproving or proving its contradictory".

To sum up, (C5) implies that, when *h* and *e* are at odds, lower values of confirmation measure $C_P(h,e)$ precisely amount to a higher degree of partial mutual inconsistency. (C6), on the other hand, implies that the positive confirmation or support from *e* to *h* is in fact nothing other than a strictly increasing function of the degree of partial inconsistency between *e* and $\neg h$. Hawthorne's (2014) aforesaid "impossibility" statement would suggest that no sensible probabilistic analysis of confirmation could satisfy such requirements, no matter how appealing they may seem. The following result, however, opens up a different scenario (see Crupi and Tentori 2013 for a proof):

Theorem 2 *(F), (C1), (C5) and (C6) if and only if $C_P(h,e)$ is a relative distance measure.*

As pointed out earlier, probabilistic measures of incremental confirmation are known to be many and diverse. Whatever the amount of pluralism that one is willing to allow for in this respect, Theorem 2 shows that a small set of properties single out relative distance measures as uniquely capturing the notion of partial entailment (and refutation). In fact, whilst all the alternatives discussed above satisfy the fundamental assumptions (F) and (C1), they demonstrably fail to capture either (C5) or (C6), thus falling within the scope of Hawthorne's (2014) negative conclusion. And yet this conclusion does not hold unrestrictedly – a genuine confirmation-theoretic generalization of logical entailment (and refutation) is possible after all.

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APPENDIX

Theorem 1. Clause (i).

(F), (C1) and (C2) if and only if $C_P(h,e)$ is a probability difference measure.

Proof

The proof provided concerns the left-to-right implication (verification of the right-to-left implication is simple).

Notice that $P(h \wedge e) = \{[P(h|e) - P(h)] + P(h)\}P(e)$. As a consequence, by (F), there exists a function j such that, for any $h,e \in L_c$ and any $P \in \mathbf{P}$, $C_P(h,e) = j[P(h|e) - P(h), P(h), P(e)]$. With no loss of generality, we will enforce probabilistic coherence and regularity by constraining the domain of j to include triplets of values (x,y,w) such that the following conditions are jointly satisfied:

- $0 < y,w < 1$;
- $x \geq -y$, by which $x + y = P(h|e) \geq 0$, and thus $P(h \wedge e) \geq 0$;
- $x \leq 1 - y$, by which $x + y = P(h|e) \leq 1$, so that $P(h \wedge e) \leq P(e)$, and thus $P(\neg h \wedge e) \geq 0$;
- $x \leq y(1/w - 1)$, by which $[(x + y)w]/y = P(e|h) \leq 1$ so that $P(h \wedge e) \leq P(h)$, and thus $P(h \wedge \neg e) \geq 0$;
- $x \geq (1 - y)(1 - 1/w)$, by which $(x + y)w = P(h \wedge e) \geq P(h) + P(e) - 1 = y + w - 1$, and thus $P(h \wedge e) + P(\neg h \wedge e) + P(h \wedge \neg e) \leq 1$.

We thus posit j: $\{(x,y,w) \in \{(-1,+1) \times (0,1)^2 \mid -y, (1 - y)(1 - 1/w) \leq x \leq y(1/w - 1), 1 - y\} \rightarrow \Re$ and denote the domain of j as D_j.

Lemma 1. For any x,y,w_1,w_2 such that $x \in (-1,+1)$, $y,w_1,w_2 \in (0,1)$, and $-y$, $(1-y)(1-1/w_1)$, $(1-y)(1-1/w_2) \leq x \leq y(1/w_1-1)$, $y(1/w_2-1)$, $1-y$, there exist $h,e_1,e_2 \in L_c$ and $P' \in \boldsymbol{P}$ such that $P'(h|e_1) - P'(h) = P'(h|e_2) - P'(h) = x$, $P'(h) = y$, $P'(e_1) = w_1$, and $P'(e_2) = w_2$.

Proof. The equalities in Lemma 1 arise from the following scheme of probability assignments:

$$P'(h \wedge e_1 \wedge e_2) = \frac{(x+y)^2 w_1 w_2}{y};$$

$$P'(h \wedge e_1 \wedge \neg e_2) = (x+y)w_1\left[1 - \frac{(x+y)w_2}{y}\right];$$

$$P'(h \wedge \neg e_1 \wedge e_2) = \left[1 - \frac{(x+y)w_1}{y}\right](x+y)w_2;$$

$$P'(h \wedge \neg e_1 \wedge \neg e_2) = \left[1 - \frac{(x+y)w_1}{y}\right]\left[1 - \frac{(x+y)w_2}{y}\right]y;$$

$$P'(\neg h \wedge e_1 \wedge e_2) = \frac{\left(1-x-y\right)^2 w_1 w_2}{(1-y)};$$

$$P'(\neg h \wedge e_1 \wedge \neg e_2) = \left(1-x-y\right)w_1\left[1 - \frac{\left(1-x-y\right)w_2}{(1-y)}\right];$$

$$P'(\neg h \wedge \neg e_1 \wedge e_2) = \left[1 - \frac{\left(1-x-y\right)w_1}{(1-y)}\right]\left(1-x-y\right)w_2;$$

$$P'(\neg h \wedge \neg e_1 \wedge \neg e_2) = \left[1 - \frac{\left(1-x-y\right)w_1}{(1-y)}\right]\left[1 - \frac{\left(1-x-y\right)w_2}{(1-y)}\right](1-y).$$

Suppose there exist (x,y,w_1), $(x,y,w_2) \in D_j$ such that $j(x,y,w_1) \neq j(x,y,w_2)$. Then, by Lemma 1 and the definition of D_j, there exist $h,e_1,e_2 \in L_c$ and $P' \in \boldsymbol{P}$ such that $P'(h|e_1) - P'(h) = P'(h|e_2) - P'(h) = x$, $P'(h) = y$, $P'(e_1) = w_1$ and $P'(e_2) = w_2$. Clearly, if the latter equalities hold, then $P'(h|e_1) = P'(h|e_2)$. Thus, there exist $h,e_1,e_2 \in L_c$ and $P' \in \boldsymbol{P}$ such that $C_{P'}(h,e_1) = j(x,y,w_1) \neq j(x,y,w_2) = C_{P'}(h,e_2)$ even if $P'(h|e_1) = P'(h|e_2)$, contradicting (C1). Conversely, (C1) implies that, for any (x,y,w_1), $(x,y,w_2) \in D_j$, $j(x,y,w_1) = j(x,y,w_2)$. So, for (C1) to hold, there must exist k such that, for any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, $C_P(h,e) = k[P(h|e) - P(h),P(h)]$ and $k(x,y) = j(x,y,w)$. We thus posit k: $\{(x,y) \in \{(-1,+1) \times (0,1)\ |-y \leq x \leq 1-y\} \rightarrow \Re$ and denote the domain of k as D_k.

Lemma 2. For any x,y_1,y_2 such that $x \in (-1,+1)$, $y_1,y_2 \in (0,1)$, $-y_1 \leq x \leq 1-y_2$, and $y_1 < y_2$, there exist $h_1,h_2,e \in L_c$ and $P'' \in \boldsymbol{P}$ such that $P''(h_1|e) - P''(h_1) = x$, $P''(h_1) = y_1$, $P''(h_1 \vee h_2) = y_2$, $P''(h_2|e) = P''(h_2)$, and $P''(h_1 \wedge h_2) = 0$.

Proof. Let $w \in (0,1)$ be given so that $w \leq y_1/(x+y_1)$, $(1-y_2)/(1-x-y_2)$ (as the latter quantities must be positive, w exists), and posit $h_2 = \neg h_1 \wedge q$, with q an atomic sentence in L_c and $q \neq h_1$. The equalities in Lemma 2 arise from the following scheme of probability assignments:

$$P''(h_1 \wedge q \wedge e) = (1/2)(x+y_1)w;$$

$$P''(\neg h_1 \wedge q \wedge e) = (y_2 - y_1)w;$$

$$P''(h_1 \wedge q \wedge \neg e) = (1/2)y_1 - (1/2)(x+y_1)w;$$

$$P''(\neg h_1 \wedge q \wedge \neg e) = (y_2 - y_1)(1-w);$$

$$P''(h_1 \wedge \neg q \wedge e) = (1/2)(x+y_1)w;$$
$$P''(\neg h_1 \wedge \neg q \wedge e) = (1 - x - y_2)w;$$
$$P''(h_1 \wedge \neg q \wedge \neg e) = (1/2)y_1 - (1/2)(x+y_1)w;$$
$$P''(\neg h_1 \wedge \neg q \wedge \neg e) = (1 - y_2) - (1 - x - y_2)w.$$

Suppose there exist (x,y_1), $(x,y_2) \in D_k$ such that $k(x,y_1) \neq k(x,y_2)$. Assume $y_1 < y_2$ with no loss of generality. Then, by Lemma 2 and the definition of D_k, there exist $h_1,h_2,e \in L_c$ and $P'' \in \boldsymbol{P}$ such that $P''(h_1|e) - P''(h_1) = x$, $P''(h_1) = y_1$, $P''(h_1 \vee h_2) = y_2$, $P''(h_2|e) = P''(h_2)$, and $P''(h_1 \wedge h_2) = 0$. By the probability calculus, if the latter equalities hold, then $P''(h_1|e) - P''(h_1) = P''(h_1 \vee h_2|e) - P''(h_1 \vee h_2) = x$. Thus, there exist $h_1,h_2,e \in L_c$ and $P'' \in \boldsymbol{P}$ such that $P''(h_1 \wedge h_2) = 0$ and $C_{P''}(h_1,e) = k(x,y_1) \neq k(x,y_2) = C_{P''}(h_1 \vee h_2,e)$ even if $P''(h_2|e) = P''(h_2)$, contradicting (C2). Conversely, (C2) implies that, for any (x,y_1), $(x,y_2) \in D_k$, $k(x,y_1) = k(x,y_2)$. So, for (C2) to hold, there must exist f such that, for any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, $C_P(h,e) = f[P(h|e) - P(h)]$ and $f(x) = k(x,y)$. We thus posit f: $(-1,+1) \rightarrow \Re$ and denote the domain of f as D_f.

Lemma 3. For any x_1,x_2 such that $x_1,x_2 \in (-1,+1)$ and $0 \leq x_1 - x_2 < 1$, there exist $h,e_1,e_2 \in L_c$ and $P''' \in \boldsymbol{P}$ such that $P'''(h|e_1) - P'''(h) = x_1$ and $P'''(h|e_2) - P'''(h) = x_2$.

Proof. Let $y,w_1,w_2 \in (0,1)$ be given so that $-x_2 \leq y \leq 1 - x_1$ (as $0 \leq x_1 - x_2 < 1$, y exists), $w_1 \leq y(x_1+y)$, $(1-y)/(1-x_1-y)$ (as the latter quantities must be positive, w_1 exists), and $w_2 \leq y(x_2+y)$, $(1-y)/(1-x_2-y)$ (as the latter quantities must be positive, w_2 exists). The equalities in Lemma 3 arise from the following scheme of probability assignments:

$$P'''(h \wedge e_1 \wedge e_2) = \frac{(x_1+y)(x_2+y)w_1 w_2}{y};$$
$$P'''(h \wedge e_1 \wedge \neg e_2) = (x_1+y)w_1\left[1 - \frac{(x_2+y)w_2}{y}\right];$$
$$P'''(h \wedge \neg e_1 \wedge e_2) = \left[1 - \frac{(x_1+y)w_1}{y}\right](x_2+y)w_2;$$
$$P'''(h \wedge \neg e_1 \wedge \neg e_2) = \left[1 - \frac{(x_1+y)w_1}{y}\right]\left[1 - \frac{(x_2+y)w_2}{y}\right]y;$$
$$P'''(\neg h \wedge e_1 \wedge e_2) = \frac{(1-x_1-y)(1-x_2-y)w_1 w_2}{(1-y)};$$
$$P'''(\neg h \wedge e_1 \wedge \neg e_2) = (1-x_1-y)w_1\left[1 - \frac{(1-x_2-y)w_2}{(1-y)}\right];$$
$$P'''(\neg h \wedge \neg e_1 \wedge e_2) = \left[1 - \frac{(1-x_1-y)w_1}{(1-y)}\right](1-x_2-y)w_2;$$
$$P'''(\neg h \wedge \neg e_1 \wedge \neg e_2) = \left[1 - \frac{(1-x_1-y)w_1}{(1-y)}\right]\left[1 - \frac{(1-x_2-y)w_2}{(1-y)}\right](1-y).$$

Let $x_1,x_2 \in D_f$ be given so that $x_1 - x_2 > 0$, i.e., $x_1 > x_2$. Let us consider two different cases. (i) First case: $x_1 - x_2 < 1$. Suppose $f(x_1) \leq f(x_2)$. Then, by Lemma 3 and the definition of D_f, there exist $h,e_1,e_2 \in L_c$ and $P''' \in \boldsymbol{P}$ such that $P'''(h|e_1) - P'''(h) = x_1$ and $P'''(h|e_2) - P'''(h) = x_2$. Clearly, if the latter equalities hold, then $P'''(h|e_1) > P'''(h|e_2)$. Thus, there exist $h,e_1,e_2 \in L_c$ and $P''' \in \boldsymbol{P}$ such that $C_{P'''}(h,e_1) = f(x_1) \leq f(x_2) = C_{P'''}(h,e_2)$ even if $P'''(h|e_1) > P'''(h|e_2)$, contradicting (C1). Conversely, (C1) implies that, for any $x_1,x_2 \in D_f$ such that $x_1 - x_2 < 1$, if $x_1 > x_2$ then $f(x_1) > f(x_2)$. (ii) Second case: $x_1 - x_2 \geq 1$. Given the definition of D_f, it is easy to show that $0 < x_1 - x_1/2 < 1$, $0 < x_1/2 - x_2/2 < 1$, and $0 < x_2/2 - x_2 < 1$ (here it is useful to note that, if $x_1 - x_2 \geq 1$, then $x_1 > 0 > x_2$). Relying on Lemma 3 as before and on the first case (i) above, we now have that (C1) implies $f(x_1) > f(x_1/2) > f(x_2/2) > f(x_2)$. Thus (C1) implies that, for any $x_1,x_2 \in D_f$ such that $x_1 - x_2 \geq 1$, if $x_1 > x_2$ then $f(x_1) > f(x_2)$. As cases (i) and (ii) are exhaustive, (C1) implies that, for any $x_1,x_2 \in D_f$, if $x_1 > x_2$ then $f(x_1) > f(x_2)$. By a similar argument, (C1) also implies that, for any $x_1,x_2 \in D_f$, if $x_1 = x_2$ then $f(x_1) = f(x_2)$. So, for (C1) to hold, it must be that, for any $h,e \in L_c$ and any $P \in \boldsymbol{P}$, $C_P(h,e) = f[P(h|e) - P(h)]$ and f is a strictly increasing function. □