Confirmation as partial entailment
A corrected proof

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January 2014

Introduction
Michael Schippers pointed out to us in personal correspondence an error in the proof of the main result in Crupi, V. and Tentori, K., "Confirmation as partial entailment: A representation theorem", Journal of Applied Logic, 11 (2013), pp. 364-372. The flaw spotted by Schippers is that Lemma 2 (p. 369) does not hold in its original formulation. In what follows, we recapitulate the proof in a corrected fashion. As a matter of fact, however, the only significant differences will concern Lemma 2 itself.

The theorem
Let $\mathcal{L}$ be a (finite) propositional language, $\mathcal{L}_c$ the set of the contingent formulae in $\mathcal{L}$ (i.e., those expressing neither logical truths nor logical falsehoods), and $\mathcal{P}$ the set of all regular probability functions that can be defined over $\mathcal{L}$ (so that, for any $\alpha \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $0 < P(\alpha) < 1$). We will posit a function $\mathcal{C} : \{\mathcal{L}_c \times \mathcal{L}_c \times \mathcal{P}\} \rightarrow \mathbb{R}$ as representing the fundamental inductive-logical relation of support or confirmation and adopt the notation $\mathcal{C}_P(h,e)$, with $e,h \in \mathcal{L}_c$ denoting the premise (or the conjunction of a collection of premises) and the conclusion of an inductive argument, respectively.

Axioms

A0 (Formality). There exists a function $g$ such that, for any $e,h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $\mathcal{C}_P(h,e) = g(P(h \land e), P(h), P(e))$. 

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A1 (Final probability incrementality). For any \( e_1, e_2, h \in \mathcal{L} \) and any \( P \in \mathcal{P} \), \( \mathcal{C}_P(h, e_1) \geq \mathcal{C}_P(h, e_2) \) if and only if \( P(h|e_1) \geq P(h|e_2) \).

A2 (Partial inconsistency). For any \( e, h \in \mathcal{L} \) and any \( P \in \mathcal{P} \), if \( P(h \land e) \leq P(h)P(e) \), then \( \mathcal{C}_P(h, e) = \mathcal{C}_P(e, h) \).

A3 (Complementarity). For any \( e, h_1, h_2 \in \mathcal{L} \) and any \( P \in \mathcal{P} \), \( \mathcal{C}_P(h_1, e) \geq \mathcal{C}_P(h_2, e) \) if and only if \( \mathcal{C}_P(\neg h_2, e) \geq \mathcal{C}_P(\neg h_1, e) \).

Theorem. A0-A3 hold if and only if there exists a strictly increasing function \( f \) such that \( \mathcal{C}_P(h, e) = f[z(h, e)] \), where

\[
  z(h, e) = \begin{cases} 
  \frac{P(h|e) - P(h)}{1 - P(h)} & \text{if } P(h|e) \geq P(h) \\
  \frac{P(h|e) - P(h)}{P(h)} & \text{if } P(h|e) < P(h)
  \end{cases}
\]

Proof

Right-to-left implication

A0. If there exists a strictly increasing function \( f \) such that \( \mathcal{C}_P(h, e) = f[z(h, e)] \), then A0 is trivially satisfied.

A1. Let \( e_1, e_2, h \in \mathcal{L} \) and \( P \in \mathcal{P} \) be given. Three classes of cases can obtain. (i) Let \( P \in \mathcal{P} \) be such that \( P(h|e_1) \geq P(h) \geq P(h|e_2) \). It is easy to verify that, for any \( e, h \in \mathcal{L} \) and any \( P \in \mathcal{P} \), \( P(h|e) \geq P(h) \) iff \( z(h, e) \geq 0 \). So we have that, for any \( e_1, e_2, h \in \mathcal{L} \), \( P(h|e_1) \geq P(h) \) iff \( z(h, e_1) \geq 0 \) and \( P(h|e_2) \geq P(h) \) iff \( z(h, e_2) \geq 0 \). It follows that, for any \( e_1, e_2, h \in \mathcal{L} \), \( z(h, e_1) \geq z(h, e_2) \) iff \( P(h|e_1) \geq P(h|e_2) \). (ii) Let \( P \in \mathcal{P} \) be such that \( P(h|e_1), P(h|e_2) \geq P(h) \). Then we have that, for any \( e_1, e_2, h \in \mathcal{L} \), \( P(h|e_1) \geq P(h|e_2) \) iff \( P(h|e_1) \leq P(h|e_2) \) iff \( 1 - \frac{P(h|e_1)}{P(h)} \leq 1 - \frac{P(h|e_2)}{P(h)} \) iff \( z(h, e_1) \leq z(h, e_2) \). (iii) Finally, let \( P \in \mathcal{P} \) be such that \( P(h|e_1), P(h|e_2) \leq P(h) \). Then we have that, for any \( e_1, e_2, h \in \mathcal{L} \), \( P(h|e_1) \leq P(h|e_2) \) iff \( P(h|e_1) \geq P(h|e_2) \) iff \( \frac{P(h|e_1)}{P(h)} \geq \frac{P(h|e_2)}{P(h)} \) iff
A2. Let $e, h \in \mathcal{L}_c$ and $P \in \mathcal{P}$ be given so that $P(h \land e) \leq P(h)P(e)$. This is equivalent to both $P(h|e) \leq P(h)$ and $P(e|h) \leq P(e)$. So we have that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h \land e) \leq P(h)P(e)$, then

$$\frac{P(h|e)}{P(h)} - 1 \geq \frac{P(h|e)}{P(h)} - 1$$

iff $z(h, e_1) \geq z(h, e_2)$. As (i)-(iii) are exhaustive, for any $e_1, e_2, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $z(h, e_1) \geq z(h, e_2)$ if and only if $P(h|e_1) \geq P(h|e_2)$. By ordinal equivalence, if there exists a strictly increasing function $f$ such that $C_P(h, e) = f[z(h, e)]$, then A1 follows.

A3. Let $e, h_1, h_2 \in \mathcal{L}_c$ and $P \in \mathcal{P}$ be given. Three classes of cases can obtain. (i) Let $P \in \mathcal{P}$ be such that $P(h_1|e) \geq P(h_1)$ and $P(h_2|e) \leq P(h_2)$. It is easy to verify that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $P(h_1|e) \geq P(h_1)$ iff $z(h, e) \geq z(h, e)$.

So we have that, for any $e, h_1, h_2, \in \mathcal{L}_c$, $P(h_1|e) \geq P(h_1)$ iff $z(h_1, e) \geq z(h_1, e)$.

(ii) Let $P \in \mathcal{P}$ be such that $P(h_1|e) = P(h_1)$ and $P(h_2|e) = P(h_2)$. Then we have that, for any $e, h_1, h_2, \in \mathcal{L}_c$, $z(h_1, e) = z(h_1, e)$ iff $z(h_2, e) = z(h_2, e)$.

(iii) Finally, let $P \in \mathcal{P}$ be such that $P(h_1|e) \leq P(h_1)$ and $P(h_2|e) = P(h_2)$. Then we have that, for any $e, h_1, h_2, \in \mathcal{L}_c$, $z(h_1, e) \leq z(h_1, e)$ if $P(h_2|e) - 1 \leq P(h_2|e) - 1$ iff $z(h_1, e) \leq z(h_1, e)$. As (i)-(iii) are exhaustive, for any $e, h_1, h_2 \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $z(h_1, e) \leq z(h_1, e)$ if and only if $z(h_1, e) \leq z(h_1, e)$. By ordinal equivalence, if there exists a strictly increasing function $f$ such that $C_P(h, e) = f[z(h, e)]$, then A3 follows.
Left-to-right implication

The case of disconfirmation \( P(h|e) \leq P(h) \)

Note that \( P(h \land e) = \frac{P(h|e)}{P(h)} P(h) P(e) \). As a consequence, by A0, there exists a function \( j \) such that, for any \( e, h \in L_c \) and any \( P \in \mathcal{P} \), \( C_P(h,e) = j[\frac{P(h|e)}{P(h)}, P(h), P(e)] \). With no loss of generality, we will convey probabilistic coherence, regularity, and disconfirmation by constraining the domain of \( j \) to include triplets of values \((x, y, w)\) such that the following conditions are jointly satisfied:

- \( 0 < y, w < 1; \)
- \( x \geq 0, \) by which \( x = \frac{P(h|e)}{P(h)} \geq 0, \) so that \( P(h|e) \geq 0, \) and thus \( P(h \land e) \geq 0; \)
- \( x \leq 1 \) (conveying disconfirmation, i.e., \( P(h|e) \leq P(h) \)), by which \( xy = P(h|e) < 1, \) so that \( P(h \land e) < P(e) \) and thus \( P(\neg h \land e) > 0, \) and \( xw = P(h|e) < 1, \) so that \( P(h \land e) < P(h) \) and thus \( P(h \land \neg e) > 0; \)
- \( x \geq \frac{y+w-1}{yw} \) (as \( y, w < 1, \) the latter quantity is necessarily lower than 1), by which \( xyw = P(h \land e) \geq P(h) + P(e) - 1 = y + w - 1, \) and thus \( P(h \land e) + P(\neg h \land e) + P(h \land \neg e) \leq 1. \)

We then posit \( j : \{(x, y, w) \in [0, 1] \times (0, 1)^2 | x \geq \frac{y+w-1}{yw}\} \rightarrow \mathbb{R} \) and denote the domain of \( j \) as \( D_j \).

**Lemma 1.** For any \( x, y, w_1, w_2 \) such that \( x \in [0, 1], y, w_1, w_2 \in (0, 1) \) and \( x \geq \frac{y+w_1-1}{yw_1}, \frac{y+w_2-1}{yw_2} \), there exist \( e_1, e_2, h \in L_c \) and \( P' \in \mathcal{P} \) such that \( \frac{P'(h|e_1)}{P'(h)} = \frac{P'(h|e_2)}{P'(h)} = x, \) \( P'(h) = y, \) \( P'(e_1) = w_1, \) and \( P'(e_2) = w_2. \)

**Proof.** The equalities in Lemma 1 arise from the following scheme of probability assignments:

\[
\begin{align*}
P'(h \land e_1 \land e_2) &= (xw_1)(xw_2)y \\
P'(h \land e_1 \land \neg e_2) &= (xw_1)(1-xw_2)y \\
P'(h \land \neg e_1 \land e_2) &= (1-xw_1)(xw_2)y \\
P'(h \land \neg e_1 \land \neg e_2) &= (1-xw_1)(1-xw_2)y \\
P'(-h \land e_1 \land e_2) &= \left(1-\frac{y}{1-y}\right)^2w_1w_2 \\
P'(-h \land e_1 \land \neg e_2) &= (1-xy)w_1[1-\left(1-\frac{1-xy}{1-y}\right)\left(1-\frac{1-xy}{1-y}\right)w_2] \\
P'(-h \land \neg e_1 \land e_2) &= [1-\left(1-\frac{1-xy}{1-y}\right)w_1](1-xy)w_2 \\
P'(-h \land \neg e_1 \land \neg e_2) &= [1-\left(1-\frac{1-xy}{1-y}\right)w_1][1-\left(1-\frac{1-xy}{1-y}\right)w_2](1-y)
\end{align*}
\]
Suppose there exist \((x, y, w_1), (x, y, w_2) \in D_j\) such that \(j(x, y, w_1) \neq j(x, y, w_2)\). Then, by Lemma 1 and the definition of \(D_j\), there exist \(e_1, e_2, h \in \mathcal{L}_c\) and \(P' \in \mathcal{P}\) such that \(P'(h|e_1) = P'(h|e_2) = x, P'(h) = y, P'(e_1) = w_1,\) and \(P'(e_2) = w_2\). Clearly, if the latter equalities hold, then \(P'(h|e_1) = P'(h|e_2)\). Thus, there exist \(e_1, e_2, h \in \mathcal{L}_c\) and \(P' \in \mathcal{P}\) such that \(C_{P'}(h, e_1) = j(x, y, w_1) \neq j(x, y, w_2) = C_{P'}(h, e_2)\) even if \(P'(h|e_1) = P'(h|e_2)\), contradicting A1. Conversely, A1 implies that, for any \((x, y, w_1), (x, y, w_2) \in D_j, j(x, y, w_1) = j(x, y, w_2)\). So, for A1 to hold, there must exist a function \(k\) such that, for any \(e, h \in \mathcal{L}_c\) and any \(P \in \mathcal{P}\), if \(P(h|e) \leq P(h)\), then \(C_{P}(h, e) = k\left[\frac{P'(h|e)}{P'(h)}\right], P(h)\) and \(k(x, y) = j(x, y, w)\). We then posit \(k : \{ (x, y) \in [0, 1] \times (0, 1) \} \rightarrow \mathbb{R}\) and denote the domain of \(k\) as \(D_k\).

**Lemma 2.** For any \(x, y, z\) such that \(x \in [0, 1], y, z \in (0, 1)\), there exist \(e, h_1, h_2 \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\) such that \(\frac{P''(h_1|e)}{P''(h_2|e)} = x, P''(h_1) = y_1,\) and \(P''(h_2) = y_2\).

**Proof.** Let \(w \in (0, 1)\) be given so that \(w < \frac{1-w_1}{1-x}, \frac{1-w_2}{1-x} (as the latter quantities must all be positive, \(w\) exists). The equalities in Lemma 2 arise from the following scheme of probability assignments:

\[
\begin{align*}
P''(h_1 \land h_2 \land e) &= x^2y_1y_2w \\
P''(h_1 \land h_2 \land \neg e) &= (1-xw)y_1y_2 \\
P''(h_1 \land \neg h_2 \land e) &= xy_1(1-xy_2)w \\
P''(h_1 \land \neg h_2 \land \neg e) &= (1-xw)y_1[1-(1-xw)y_2] \\
P''(\neg h_1 \land h_2 \land e) &= (1-xy_1)y_2w \\
P''(\neg h_1 \land h_2 \land \neg e) &= [1-(1-xw)y_1](1-xw)y_2 \\
P''(\neg h_1 \land \neg h_2 \land e) &= (1-xy_1)(1-xy_2)w \\
P''(\neg h_1 \land \neg h_2 \land \neg e) &= [1-(1-xw)y_1][1-(1-xw)y_2](1-w)
\end{align*}
\]

Suppose there exist \((x, y_1), (x, y_2) \in D_k\) such that \(k(x, y_1) \neq k(x, y_2)\). Then, by Lemma 2 and the definition of \(D_k\), there exist \(e, h_1, h_2 \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\) such that \(\frac{P''(h_1|e)}{P''(h_2|e)} = x, P''(h_1) = y_1, P''(h_2) = y_2,\) and \(P''(e) = w\). By the probability calculus, if the latter equalities hold, then \(P''(h_1 \land e) \leq P''(h_1)P''(e), P''(h_2 \land e) \leq P''(h_2)P''(e),\) and moreover \(\frac{P''(e|h_1)}{P''(e)} = \frac{P''(h_2|e)}{P''(e)} = x\). Thus, there exist \(e, h_1, h_2 \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\) such that
\(\mathcal{P}\) such that either \(C_{P''}(h_1, e) = k(x, y_1) \neq k(x, w) = C_{P''}(e, h_1)\) even if \(P''(h_1 \land e) \leq P''(h_1)P''(e)\), or \(C_{P''}(h_2, e) = k(x, y_2) \neq k(x, w) = C_{P''}(e, h_2)\) even if \(P''(h_2 \land e) \leq P''(h_2)P''(e)\), contradicting A2. Conversely, A2 implies that, for any \((x, y_1), (x, y_2) \in D_k, k(x, y_1) = k(x, y_2)\). So, for A2 to hold, there must exist a function \(s\) such that, for any \(e, h \in \mathcal{L}_c\) and any \(P \in \mathcal{P}\), if \(P(h|e) \leq P(h)\), then \(C_P(h, e) = m \left[ \frac{P(h|e)}{P(h)} \right] \) and \(m(x) = k(x, y)\). We then posit \(m : [0, 1] \to \mathbb{R}\) and denote the domain of \(m\) as \(D_m\).

**Lemma 3.** For any \(x_1, x_2 \in [0, 1]\), there exist \(e_1, e_2, h \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\) such that \(P''(h \land e_1 \land e_2) = x_1\) and \(P''(h \land e_1 \land -e_2) = x_2\).

**Proof.** Let \(y, w_1, w_2 \in (0, 1)\) be given so that \(w_1 < \frac{1-y}{1-xy}\) (as the latter quantity must be positive, \(w_1\) exists) and \(w_2 < \frac{1-y}{1-xy}\) (as the latter quantity must be positive, \(w_2\) exists). The equalities in Lemma 3 arise from the following scheme of probability assignments:

\[
\begin{align*}
P''(h \land e_1 \land e_2) &= (x_1w_1)(x_2w_2)y \\
P''(h \land e_1 \land -e_2) &= (x_1w_1)(1 - x_2w_2)y \\
P''(h \land -e_1 \land e_2) &= (1 - x_1w_1)(x_2w_2)y \\
P''(h \land -e_1 \land -e_2) &= (1 - x_1w_1)(1 - x_2w_2)y \\
P''(-h \land e_1 \land e_2) &= (1-y)(1-x_2y)w_1w_2 \\
P''(-h \land e_1 \land -e_2) &= (1 - x_1y)w_1[1 - \frac{(1-x_2y)w_2}{1-y}] \\
P''(-h \land -e_1 \land e_2) &= [1 - \frac{(1-x_1y)w_1}{1-y}][1 - x_2y]w_2 \\
P''(-h \land -e_1 \land -e_2) &= [1 - \frac{(1-x_1y)w_1}{1-y}][1 - \frac{(1-x_2y)w_2}{1-y}](1-y)
\end{align*}
\]

Suppose there exist \(x_1, x_2 \in D_m\) such that \(x_1 > x_2\) and \(m(x_1) \leq m(x_2)\). Then, by Lemma 3 and the definition of \(D_m\), there exist \(e_1, e_2, h \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\) such that \(P''(h|e_1) = x_1\) and \(P''(h|e_2) = x_2\). Clearly, if the latter equalities hold, then \(P''(h|e_1) > P''(h|e_2)\). Thus, there exist \(e_1, e_2, h \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\) such that \(C_{P''}(h, e_1) = m(x_1) \leq m(x_2) = C_{P''}(h, e_2)\) even if \(P''(h|e_1) > P''(h|e_2)\), contradicting A1. Conversely, A1 implies that, for any \(x_1, x_2 \in D_m\), if \(x_1 > x_2\), then \(m(x_1) > m(x_2)\). By a similar argument, A1 also implies that, for any \(x_1, x_2 \in D_m\), if \(x_1 = x_2\), then \(m(x_1) = m(x_2)\). So, for A1 to hold, it must be that, for any \(e, h \in \mathcal{L}_c\) and any \(P \in \mathcal{P}\), if \(P(h|e) \leq P(h)\), then \(C_P(h, e) = m \left[ \frac{P(h|e)}{P(h)} \right] \) and \(m\) is a strictly increasing function.
The case of confirmation \((P(h|e) > P(h))\)

Note that \(P(h \land e) = [1 - \frac{P(-h|e)}{P(-h)}]P(e)\) and \(P(h) = 1 - P(-h)\). As a consequence, by A0, there exists a function \(r\) such that, for any \(e, h \in \mathcal{L}_c\) and any \(P \in \mathcal{P}\), \(\mathcal{C}_P(h, e) = r\left[\frac{P(-h|e)}{P(-h)}, P(-h), P(e)\right]\). With no loss of generality, we will convey probabilistic coherence, regularity, and confirmation by constraining the domain of \(r\) to include triplets of values \((x, y, w)\) such that the following conditions are jointly satisfied:

- \(0 < y, w < 1\);
- \(x \geq 0\), by which \(x = \frac{P(-h|e)}{P(-h)} \geq 0\), so that \(P(-h|e) \geq 0\), and thus \(P(-h \land e) \geq 0\);
- \(x < 1\) (conveying confirmation, i.e., \(P(h|e) > P(h)\)), by which \(xy = P(-h|e) < 1\), so that \(P(-h \land e) < P(e)\) and thus \(P(h \land e) > 0\), and \(xw = P(e|\neg h) < 1\), so that \(P(-h \land e) < P(-h)\) and thus \(P(-h \land \neg e) > 0\);
- \(x \geq \frac{y + w - 1}{yw}\) (as \(y, w < 1\), the latter quantity is necessarily lower than 1), by which \(xyw = P(-h \land e) \geq P(-h) + P(e) - 1 = y + w - 1\), and thus \(P(-h \land e) + P(h \land e) + P(-h \land \neg e) \leq 1\).

We then posit \(r : \{(x, y, w) \in [0, 1) \times (0, 1)^2 \mid x \geq \frac{y + w - 1}{yw}\} \to \mathbb{R}\) and denote the domain of \(r\) as \(D_r\).

Lemma 4. For any \(x, y, w_1, w_2\) such that \(x \in [0, 1), y, w_1, w_2 \in (0, 1)\) and \(x \geq \frac{y + w_1 - 1}{yaw_1}, \frac{y + w_2 - 1}{yaw_2}\), there exist \(e_1, e_2, h \in \mathcal{L}_c\) and \(P' \in \mathcal{P}\) such that \(\frac{P'(h|e_1)}{P'(h|\neg h)} = \frac{P'(h|e_2)}{P'(h|\neg h)} = x\), \(P'(-h) = y\), \(P'(e_1) = w_1\), and \(P'(e_2) = w_2\).

Proof. The equalities in Lemma 4 arise from the following scheme of probability assignments:

\[
\begin{align*}
P'(h \land e_1 \land e_2) &= \frac{(1-xy)^2w_1w_2}{1-y} \\
P'(h \land e_1 \land \neg e_2) &= (1 - xy)w_1[1 - \frac{(1-xy)w_2}{1-y}] \\
P'(h \land \neg e_1 \land e_2) &= [1 - \frac{(1-xy)w_1}{1-y}][1 - xy]w_2 \\
P'(h \land \neg e_1 \land \neg e_2) &= [1 - \frac{(1-xy)w_1}{1-y}[1 - \frac{(1-xy)w_2}{1-y}](1-y) \\
P'(\neg h \land e_1 \land e_2) &= (xw_1)(xw_2)y \\
P'(\neg h \land e_1 \land \neg e_2) &= (xw_1)(1-xw_2)y \\
P'(\neg h \land \neg e_1 \land e_2) &= (1 - xw_1)(xw_2)y \\
P'(\neg h \land \neg e_1 \land \neg e_2) &= (1 - xw_1)(1-xw_2)y
\end{align*}
\]
Suppose there exist \((x, y, w_1), (x, y, w_2)\) \(\in D_r\) such that \(r(x, y, w_1) \neq r(x, y, w_2)\). Then, by Lemma 4 and the definition of \(D_r\), there exist \(e_1, e_2, h \in \mathcal{L}_c\) and \(P' \in \mathcal{P}\) such that \(P'(-h|e_1) = P'(-h|e_2) = x\), \(P'(-h) = y\), \(P'(e_1) = w_1\), and \(P'(e_2) = w_2\). By the probability calculus, if the latter equalities hold, then \(P'(h|e_1) = P'(h|e_2)\). Thus, there exist \(e_1, e_2, h \in \mathcal{L}_c\) and \(P' \in \mathcal{P}\) such that \(C_{P'}(h, e_1) = r(x, y, w_1) \neq r(x, y, w_2) = C_{P'}(h, e_2)\) even if \(P'(h|e_1) = P'(h|e_2)\), contradicting A1. Conversely, A1 implies that, for any \((x, y, w_1), (x, y, w_2) \in D_r\), \(r(x, y, w_1) = r(x, y, w_2)\). So, for A1 to hold, there must exist a function \(s\) such that, for any \(e, h \in \mathcal{L}_c\) and any \(P \in \mathcal{P}\), if \(P(h|e) > P(h|e)\), then \(C_{P}(h, e) = s(P(h|e), P(h|e))\) and \(s(x, y) = r(x, y, w)\).

We then posit \(s : \{(x, y) \in [0, 1) \times (0, 1)\} \to \mathbb{R}\) and denote the domain of \(s\) as \(D_s\).

**Lemma 5.** For any \(x, y_1, y_2\) such that \(x \in [0, 1), y_1, y_2 \in (0, 1)\), there exist \(e, h_1, h_2 \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\) such that \(P''(-h_1) = y_1\), and \(P''(-h_2) = y_2\).

**Proof.** Let \(w \in (0, 1)\) be given so that \(w \leq \frac{1-y_1}{1-x_1}, \frac{1-y_2}{1-x_2}\) (as the latter quantities must all be positive, \(w\) exists). The equalities in Lemma 5 arise from the following scheme of probability assignments:

\[
\begin{align*}
P''(h_1 \wedge h_2 \wedge e) &= (1 - x_1)(1 - x_2)w \\
P''(h_1 \wedge h_2 \wedge \neg e) &= [1 - (1 - x_1)w][1 - (1 - x_2)w](1 - w) \\
P''(h_1 \wedge \neg h_2 \wedge e) &= (1 - x_1)(x_2)w \\
P''(h_1 \wedge \neg h_2 \wedge \neg e) &= [1 - (1 - x_1)w](1 - x_2)w \\
P''(-h_1 \wedge h_2 \wedge e) &= (x_1)(1 - x_2)w \\
P''(-h_1 \wedge h_2 \wedge \neg e) &= (x_1)(1 - x_2)w \\
P''(-h_1 \wedge \neg h_2 \wedge e) &= (x_1)(x_2)w \\
P''(-h_1 \wedge \neg h_2 \wedge \neg e) &= \frac{(1 - x_1)(1 - x_2)w}{1 - w} \\
\end{align*}
\]

Suppose there exist \((x, y_1), (x, y_2) \in D_s\) such that \(s(x, y_1) \neq s(x, y_2)\). Then, by Lemma 5 and the definition of \(D_s\), there exist \(e, h_1, h_2 \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\) such that \(P''(-h_1|e) = P''(-h_2|e) = x\), \(P''(-h_1) = y_1\), \(P''(-h_2) = y_2\), and \(P''(e) = w\). If the latter equalities hold, then \(C_{P''}(-h_1, e) = mP''(-h_1|e) = mP''(-h_2|e) = C_{P''}(-h_2, e)\). Thus, there exist \(e, h_1, h_2 \in \mathcal{L}_c\) and \(P'' \in \mathcal{P}\).
such that \( C_{P'}(h_1, e) = s(x, y_1) \neq s(x, y_2) = C_{P'}(h_2, e) \) even if \( C_{P'}(-h_1, e) = C_{P'}(-h_2, e) \), contradicting A3. Conversely, A3 implies that, for any \((x, y_1), (x, y_2) \in \mathcal{D}_s\), \( s(x, y_1) = s(x, y_2) \). So, for A3 to hold, there must exist a function \( t \) such that, for any \( e, h \in \mathcal{L}_c \) and any \( P \in \mathcal{P} \), if \( P(h|e) > P(h) \), then \( C_P(h, e) = t\left[ \frac{P(-h|e)}{P(-h)} \right] \) and \( t(x) = s(x, y) \). We then posit \( t : [0, 1) \rightarrow \mathbb{R} \) and denote the domain of \( t \) as \( \mathcal{D}_t \).

**Lemma 6.** For any \( x_1, x_2 \in [0, 1) \), there exist \( e_1, e_2, h \in \mathcal{L}_c \) and \( P'' \in \mathcal{P} \) such that \( \frac{P''(h|e_1)}{P''(-h|e_1)} = x_1 \) and \( \frac{P''(h|e_2)}{P''(-h|e_2)} = x_2 \).

**Proof.** Let \( y, w_1, w_2 \in (0, 1) \) be given so that \( w_1 < \frac{1-y}{1-y} \) (as the latter quantity must be positive, \( w_1 \) exists) and \( w_2 < \frac{1-y}{1-y} \) (as the latter quantity must be positive, \( w_2 \) exists). The equalities in Lemma 6 arise from the following scheme of probability assignments:

\[
\begin{align*}
P''(h \land e_1 \land e_2) &= (1-x_1y)(1-x_2y)w_1w_2 \\
\frac{P''(h \land e_1 \land \neg e_2)}{P''(-h \land e_1 \land e_2)} &= (1 - x_1y)w_1[1 - \frac{(1-x_2y)w_2}{1-y}] \\
\frac{P''(h \land \neg e_1 \land e_2)}{P''(-h \land e_1 \land e_2)} &= [1 - \frac{(1-x_1y)w_1}{1-y}][1 - \frac{(1-x_2y)w_2}{1-y}](1-y) \\
\frac{P''(h \land \neg e_1 \land e_2)}{P''(-h \land e_1 \land e_2)} &= (x_1w_1)(x_2w_2)y \\
\frac{P''(h \land e_1 \land \neg e_2)}{P''(-h \land e_1 \land e_2)} &= (x_1w_1)(1 - x_2w_2)y \\
\frac{P''(h \land \neg e_1 \land \neg e_2)}{P''(-h \land e_1 \land \neg e_2)} &= (1 - x_1w_1)(x_2w_2)y \\
\frac{P''(h \land \neg e_1 \land e_2)}{P''(-h \land e_1 \land e_2)} &= (1 - x_1w_1)(1 - x_2w_2)y
\end{align*}
\]

Suppose there exist \( x_1, x_2 \in \mathcal{D}_t \) such that \( x_1 > x_2 \) and \( t(x_1) \geq t(x_2) \). Then, by Lemma 6 and the definition of \( \mathcal{D}_t \), there exist \( e_1, e_2, h \in \mathcal{L}_c \) and \( P'' \in \mathcal{P} \) such that \( \frac{P''(h|e_1)}{P''(-h|e_1)} = x_1 \) and \( \frac{P''(h|e_2)}{P''(-h|e_2)} = x_2 \). By the probability calculus, if the latter equalities hold, then \( P''(h|e_1) < P''(h|e_2) \). Thus, there exist \( e_1, e_2, h \in \mathcal{L}_c \) and \( P'' \in \mathcal{P} \) such that \( C_{P''}(h, e_1) = t(x_1) \geq t(x_2) = C_{P''}(h, e_2) \) even if \( P''(h|e_1) < P''(h|e_2) \), contradicting A1. Conversely, A1 implies that, for any \( x_1, x_2 \in \mathcal{D}_t \), if \( x_1 > x_2 \), then \( t(x_1) < t(x_2) \). By a similar argument, A1 also implies that, for any \( x_1, x_2 \in \mathcal{D}_t \), if \( x_1 = x_2 \), then \( t(x_1) = t(x_2) \). So, for A1 to hold, it must be that, for any \( e, h \in \mathcal{L}_c \) and any \( P \in \mathcal{P} \), if \( P(h|e) > P(h) \), then \( C_P(h, e) = t\left[ \frac{P(-h|e)}{P(-h)} \right] \) and \( t \) is a strictly decreasing function.
Summing up, if A0-A3 hold, then for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, (i) in case $P(h|e) \leq P(h)$, $C_P(h,e) = m \left[ \frac{P(h|e)}{P(h)} \right]$ and $m$ is a strictly increasing function, thus $C_P(h,e)$ is a strictly increasing function of $z(h,e)$, and (ii) in case $P(h|e) > P(h)$, $C_P(h,e) = t \left[ \frac{P(-h|e)}{P(-h)} \right]$ and $t$ is a strictly decreasing function, thus $C_P(h,e)$ is again a strictly increasing function of $z(h,e)$. As (i)-(ii) are exhaustive, for A0-A3 to hold, it must be that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $C_P(h,e) = f[z(h,e)]$ and $f$ is a strictly increasing function.