Confirmation as partial entailment
A corrected proof

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Introduction

Michael Schippers pointed out to us in personal correspondence an error in the proof of the main result in Crupi, V. and Tentori, K., "Confirmation as partial entailment: A representation theorem", *Journal of Applied Logic*, 11 (2013), pp. 364-372. The flaw spotted by Schippers is that Lemma 2 (p. 369) does not hold in its original formulation. In what follows, we recapitulate the proof in a corrected fashion. As a matter of fact, however, the only significant differences will concern Lemma 2 itself.

The theorem

Let $\mathcal{L}$ be a (finite) propositional language, $\mathcal{L}_c$ the set of the contingent formulae in $\mathcal{L}$ (i.e., those expressing neither logical truths nor logical falsehoods), and $\mathcal{P}$ the set of all *regular* probability functions that can be defined over $\mathcal{L}$ (so that, for any $\alpha \in \mathcal{L}_c$ and any $P \in \mathcal{P}, 0 < P(\alpha) < 1$). We will posit a function $\mathcal{C} : \{\mathcal{L}_c \times \mathcal{L}_c \times \mathcal{P}\} \rightarrow \Re$ as representing the fundamental inductive-logical relation of support or confirmation and adopt the notation $\mathcal{C}_P(h, e)$, with $e, h \in \mathcal{L}_c$ denoting the premise (or the conjunction of a collection of premises) and the conclusion of an inductive argument, respectively.

Axioms

A0 (*Formality*). There exists a function g such that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $\mathcal{C}_P(h, e) = g(P(h \wedge e), P(h), P(e))$.

A1 (*Final probability incrementality*). For any $e_1, e_2, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $\mathcal{C}_P(h, e_1) \gtreqless \mathcal{C}_P(h, e_2)$ if and only if $P(h|e_1) \gtreqless P(h|e_2)$.

A2 (*Partial inconsistency*). For any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h \wedge e) \leq P(h)P(e)$, then $\mathcal{C}_P(h, e) = \mathcal{C}_P(e, h)$.

A3 (*Complementarity*). For any $e, h_1, h_2 \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $\mathcal{C}_P(h_1, e) \gtreqless \mathcal{C}_P(h_2, e)$ if and only if $\mathcal{C}_P(\neg h_1, e) \lesseqgtr \mathcal{C}_P(\neg h_2, e)$.

Theorem. A0-A3 hold if and only if there exists a strictly increasing function f such that $\mathcal{C}_P(h, e) = f[z(h, e)]$, where

$$z(h,e) = \begin{cases} \frac{P(h|e)-P(h)}{1-P(h)} & \text{if } P(h|e) \geq P(h) \\ \frac{P(h|e)-P(h)}{P(h)} & \text{if } P(h|e) < P(h) \end{cases}$$

Proof

Right-to-left implication

A0. If there exists a strictly increasing function f such that $\mathcal{C}_P(h, e) = f[z(h, e)]$, then A0 is trivially satisfied.

A1. Let $e_1, e_2, h \in \mathcal{L}_c$ and $P \in \mathcal{P}$ be given. Three classes of cases can obtain. (i) Let $P \in \mathcal{P}$ be such that $P(h|e_1) \gtreqless P(h) \gtreqless P(h|e_2)$. It is easy to verify that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $P(h|e) \gtreqless P(h)$ iff $z(h, e) \gtreqless 0$. So we have that, for any $e_1, e_2, h \in \mathcal{L}_c$, $P(h|e_1) \gtreqless P(h)$ iff $z(h, e_1) \gtreqless 0$ and $P(h|e_2) \lesseqgtr P(h)$ iff $z(h, e_2) \lesseqgtr 0$. It follows that, for any $e_1, e_2, h \in \mathcal{L}_c$, $z(h, e_1) \gtreqless z(h, e_2)$ iff $P(h|e_1) \gtreqless P(h|e_2)$. (ii) Let $P \in \mathcal{P}$ be such that $P(h|e_1), P(h|e_2) \geq P(h)$. Then we have that, for any $e_1, e_2, h \in \mathcal{L}_c$, $P(h|e_1) \gtreqless P(h|e_2)$ iff $P(\neg h|e_1) \lesseqgtr P(\neg h|e_2)$ iff $\frac{P(\neg h|e_1)}{P(\neg h)} \lesseqgtr \frac{P(\neg h|e_2)}{P(\neg h)}$ iff $1-\frac{P(\neg h|e_1)}{P(\neg h)} \gtreqless 1-\frac{P(\neg h|e_2)}{P(\neg h)}$ iff $z(h, e_1) \gtreqless z(h, e_2)$. (iii) Finally, let $P \in \mathcal{P}$ be such that $P(h|e_1), P(h|e_2) \leq P(h)$. Then we have that, for any $e_1, e_2, h \in \mathcal{L}_c$, $P(h|e_1) \gtreqless P(h|e_2)$ iff $\frac{P(h|e_1)}{P(h)} \gtreqless \frac{P(h|e_2)}{P(h)}$ iff

$\frac{P(h|e_1)}{P(h)} - 1 \gtreqqless \frac{P(h|e_2)}{P(h)} - 1$ iff $z(h, e_1) \gtreqqless z(h, e_2)$. As (i)-(iii) are exhaustive, for any $e_1, e_2, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $z(h, e_1) \gtreqqless z(h, e_2)$ if and only if $P(h|e_1) \gtreqqless P(h|e_2)$. By ordinal equivalence, if there exists a strictly increasing function f such that $\mathcal{C}_P(h, e) = f[z(h, e)]$, then A1 follows.

A2. Let $e, h \in \mathcal{L}_c$ and $P \in \mathcal{P}$ be given so that $P(h \wedge e) \leq P(h)P(e)$. This is equivalent to both $P(h|e) \leq P(h)$ and $P(e|h) \leq P(e)$. So we have that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h \wedge e) \leq P(h)P(e)$, then $\frac{P(h|e)}{P(h)} = \frac{P(e|h)}{P(e)}$ iff $\frac{P(h|e)}{P(h)} - 1 = \frac{P(e|h)}{P(e)} - 1$ iff $z(h, e) = z(e, h)$. By ordinal equivalence, if there exists a strictly increasing function f such that $\mathcal{C}_P(h, e) = f[z(h, e)]$, then A2 follows.

A3. Let $e, h_1, h_2, \in \mathcal{L}_c$ and $P \in \mathcal{P}$ be given. Three classes of cases can obtain. (i) Let $P \in \mathcal{P}$ be such that $P(h_1|e) \gtreqqless P(h_1)$ and $P(h_2|e) \lesseqqgtr P(h_2)$. It is easy to verify that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $P(h|e) \gtreqqless P(h)$ iff $z(h, e) \gtreqqless 0$ iff $P(\neg h|e) \lesseqqgtr P(\neg h)$ iff $z(\neg h, e) \lesseqqgtr 0$. So we have that, for any $e, h_1, h_2, \in \mathcal{L}_c$, $P(h_1|e) \gtreqqless P(h_1)$ iff $z(h_1, e) \gtreqqless 0$ iff $P(\neg h_1|e) \lesseqqgtr P(\neg h_1)$ iff $z(\neg h_1, e) \lesseqqgtr 0$ and $P(h_2|e) \gtreqqless P(h_2)$ iff $z(h_2, e) \gtreqqless 0$ iff $P(\neg h_2|e) \lesseqqgtr P(\neg h_2)$ iff $z(\neg h_2, e) \lesseqqgtr 0$. It follows that, for any $e, h_1, h_2, \in \mathcal{L}_c$, $z(h_1, e) \gtreqqless z(h_2, e)$ iff $z(\neg h_1, e) \lesseqqgtr z(\neg h_2, e)$. (ii) Let $P \in \mathcal{P}$ be such that $P(h_1|e) \geq P(h_1)$ and $P(h_2|e) \geq P(h_2)$. Then we have that, for any $e, h_1, h_2, \in \mathcal{L}_c$, $z(h_1, e) \gtreqqless z(h_2, e)$ iff $1 - \frac{P(\neg h_1|e)}{P(\neg h_1)} \gtreqqless 1 - \frac{P(\neg h_2|e)}{P(\neg h_2)}$ iff $\frac{P(\neg h_1|e)}{P(\neg h_1)} \lesseqqgtr \frac{P(\neg h_2|e)}{P(\neg h_2)}$ iff $\frac{P(\neg h_1|e)}{P(\neg h_1)} - 1 \lesseqqgtr \frac{P(\neg h_2|e)}{P(\neg h_2)} - 1$ iff $z(\neg h_1, e) \lesseqqgtr z(\neg h_2, e)$. (iii) Finally, let $P \in \mathcal{P}$ be such that $P(h_1|e) \leq P(h_1)$ and $P(h_2|e) \leq P(h_2)$. Then we have that, for any $e, h_1, h_2, \in \mathcal{L}_c$, $z(h_1, e) \gtreqqless z(h_2, e)$ iff $\frac{P(\neg h_1|e)}{P(\neg h_1)} - 1 \gtreqqless \frac{P(\neg h_2|e)}{P(\neg h_2)} - 1$ iff $\frac{P(\neg h_1|e)}{P(\neg h_1)} \gtreqqless \frac{P(\neg h_2|e)}{P(\neg h_2)}$ iff $1 - \frac{P(\neg h_1|e)}{P(\neg h_1)} \lesseqqgtr 1 - \frac{P(\neg h_2|e)}{P(\neg h_2)}$ iff $z(\neg h_1, e) \lesseqqgtr z(\neg h_2, e)$. As (i)-(iii) are exhaustive, for any $e, h_1, h_2 \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $z(h, e_1) \gtreqqless z(h, e_2)$ if and only if $z(\neg h_1, e) \lesseqqgtr z(\neg h_2, e)$. By ordinal equivalence, if there exists a strictly increasing function f such that $\mathcal{C}_P(h, e) = f[z(h, e)]$, then A3 follows.

Left-to-right implication

The case of disconfirmation ($P(h|e) \leq P(h)$)

Note that $P(h \wedge e) = \frac{P(h|e)}{P(h)} P(h)P(e)$. As a consequence, by A0, there exists a function j such that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $\mathcal{C}_P(h, e) = j[\frac{P(h|e)}{P(h)}, P(h), P(e)]$. With no loss of generality, we will convey probabilistic coherence, regularity, and disconfirmation by constraining the domain of j to include triplets of values (x, y, w) such that the following conditions are jointly satisfied:

- $0 < y, w < 1$;
- $x \geq 0$, by which $x = \frac{P(h|e)}{P(h)} \geq 0$, so that $P(h|e) \geq 0$, and thus $P(h \wedge e) \geq 0$;
- $x \leq 1$ (conveying disconfirmation, i.e., $P(h|e) \leq P(h)$), by which $xy = P(h|e) < 1$, so that $P(h \wedge e) < P(e)$ and thus $P(\neg h \wedge e) > 0$, and $xw = P(e|h) < 1$, so that $P(h \wedge e) < P(h)$ and thus $P(h \wedge \neg e) > 0$;
- $x \geq \frac{y+w-1}{yw}$ (as $y, w < 1$, the latter quantity is necessarily lower than 1), by which $xyw = P(h \wedge e) \geq P(h) + P(e) - 1 = y + w - 1$, and thus $P(h \wedge e) + P(\neg h \wedge e) + P(h \wedge \neg e) \leq 1$.

We then posit $j : \{(x, y, w) \in [0, 1] \times (0, 1)^2 \mid x \geq \frac{y+w-1}{yw}\} \rightarrow \Re$ and denote the domain of j as D_j.

Lemma 1. For any x, y, w_1, w_2 such that $x \in [0, 1], y, w_1, w_2 \in (0, 1)$ and $x \geq \frac{y+w_1-1}{yw_1}, \frac{y+w_2-1}{yw_2}$, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P' \in \mathcal{P}$ such that $\frac{P'(h|e_1)}{P'(h)} = \frac{P'(h|e_2)}{P'(h)} = x, P'(h) = y, P'(e_1) = w_1$, and $P'(e_2) = w_2$.

Proof. The equalities in Lemma 1 arise from the following scheme of probability assignments:

$$
\begin{array}{ll}
P'(h \wedge e_1 \wedge e_2) & = (xw_1)(xw_2)y \\
P'(h \wedge e_1 \wedge \neg e_2) & = (xw_1)(1 - xw_2)y \\
P'(h \wedge \neg e_1 \wedge e_2) & = (1 - xw_1)(xw_2)y \\
P'(h \wedge \neg e_1 \wedge \neg e_2) & = (1 - xw_1)(1 - xw_2)y \\
P'(\neg h \wedge e_1 \wedge e_2) & = \frac{(1-xy)^2 w_1 w_2}{1-y} \\
P'(\neg h \wedge e_1 \wedge \neg e_2) & = (1 - xy)w_1[1 - \frac{(1-xy)w_2}{1-y}] \\
P'(\neg h \wedge \neg e_1 \wedge e_2) & = [1 - \frac{(1-xy)w_1}{1-y}](1 - xy)w_2 \\
P'(\neg h \wedge \neg e_1 \wedge \neg e_2) & = [1 - \frac{(1-xy)w_1}{1-y}][1 - \frac{(1-xy)w_2}{1-y}](1 - y)
\end{array}
$$

Suppose there exist $(x, y, w_1), (x, y, w_2) \in D_j$ such that $j(x, y, w_1) \neq j(x, y, w_2)$. Then, by Lemma 1 and the definition of D_j, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P' \in \mathcal{P}$ such that $\frac{P'(h|e_1)}{P'(h)} = \frac{P'(h|e_2)}{P'(h)} = x, P'(h) = y, P'(e_1) = w_1$, and $P'(e_2) = w_2$. Clearly, if the latter equalities hold, then $P'(h|e_1) = P'(h|e_2)$. Thus, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P' \in \mathcal{P}$ such that $\mathcal{C}_{P'}(h, e_1) = j(x, y, w_1) \neq j(x, y, w_2) = C_{P'}(h, e_2)$ even if $P'(h|e_1) = P'(h|e_2)$, contradicting A1. Conversely, A1 implies that, for any $(x, y, w_1), (x, y, w_2) \in D_j$, $j(x, y, w_1) = j(x, y, w_2)$. So, for A1 to hold, there must exist a function k such that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h|e) \leq P(h)$, then $\mathcal{C}_P(h, e) = k[\frac{P(h|e)}{P(h)}, P(h)]$ and $k(x, y) = j(x, y, w)$. We then posit $k : \{(x, y) \in [0, 1] \times (0, 1)\} \to \Re$ and denote the domain of k as D_k.

Lemma 2. For any x, y_1, y_2 such that $x \in [0, 1], y_1, y_2 \in (0, 1)$, there exist $e, h_1, h_2 \in \mathcal{L}_c$ and $P'' \in \mathcal{P}$ such that $\frac{P''(h_1|e)}{P''(h_1)} = \frac{P''(h_2|e)}{P'(h_2)} = x, P''(h_1) = y_1$, and $P''(h_2) = y_2$.

Proof. Let $w \in (0, 1)$ be given so that $w < \frac{1-y_1}{1-xy_1}, \frac{1-y_2}{1-xy_2}$ (as the latter quantities must all be positive, w exists). The equalities in Lemma 2 arise from the following scheme of probability assignments:

$$
\begin{aligned}
P''(h_1 \wedge h_2 \wedge e) &= x^2 y_1 y_2 w \\
P''(h_1 \wedge h_2 \wedge \neg e) &= \tfrac{(1-xw)^2 y_1 y_2}{1-w} \\
P''(h_1 \wedge \neg h_2 \wedge e) &= x y_1 (1 - x y_2) w \\
P''(h_1 \wedge \neg h_2 \wedge \neg e) &= (1 - xw) y_1 [1 - \tfrac{(1-xw)y_2}{1-w}] \\
P''(\neg h_1 \wedge h_2 \wedge e) &= (1 - x y_1) x y_2 w \\
P''(\neg h_1 \wedge h_2 \wedge \neg e) &= [1 - \tfrac{(1-xw)y_1}{1-w}](1 - xw) y_2 \\
P''(\neg h_1 \wedge \neg h_2 \wedge e) &= (1 - x y_1)(1 - x y_2) w \\
P''(\neg h_1 \wedge \neg h_2 \wedge \neg e) &= [1 - \tfrac{(1-xw)y_1}{1-w}][1 - \tfrac{(1-xw)y_1}{1-w}](1 - w)
\end{aligned}
$$

Suppose there exist $(x, y_1), (x, y_2) \in D_k$ such that $k(x, y_1) \neq k(x, y_2)$. Then, by Lemma 2 and the definition of D_k, there exist $e, h_1, h_2 \in \mathcal{L}_c$ and $P'' \in \mathcal{P}$ such that $\frac{P''(h_1|e)}{P''(h_1)} = \frac{P''(h_2|e)}{P''(h_2)} = x$, $P''(h_1) = y_1$, $P''(h_2) = y_2$, and $P''(e) = w$. By the probability calculus, if the latter equalities hold, then $P''(h_1 \wedge e) \leq P''(h_1)P''(e)$, $P''(h_2 \wedge e) \leq P''(h_2)P''(e)$, and moreover $\frac{P''(e|h_1)}{P''(e)} = \frac{P''(e|h_2)}{P''(e)} = x$. Thus, there exist $e, h_1, h_2 \in \mathcal{L}_c$ and $P'' \in$

$\mathcal{P}$ such that either $\mathcal{C}_{P''}(h_1,e) = k(x,y_1) \neq k(x,w) = C_{P''}(e,h_1)$ even if $P''(h_1 \wedge e) \leq P''(h_1)P''(e)$, or $\mathcal{C}_{P''}(h_2,e) = k(x,y_2) \neq k(x,w) = C_{P''}(e,h_2)$ even if $P''(h_2 \wedge e) \leq P''(h_2)P''(e)$, contradicting A2. Conversely, A2 implies that, for any $(x,y_1),(x,y_2) \in D_k$, $k(x,y_1) = k(x,y_2)$. So, for A2 to hold, there must exist a function m such that, for any $e,h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h|e) \leq P(h)$, then $\mathcal{C}_P(h,e) = m[\frac{P(h|e)}{P(h)}]$ and $m(x) = k(x,y)$. We then posit $m : [0,1] \to \Re$ and denote the domain of m as D_m.

Lemma 3. For any $x_1, x_2 \in [0,1]$, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P''' \in \mathcal{P}$ such that $\frac{P'''(h|e_1)}{P'''(h)} = x_1$ and $\frac{P'''(h|e_2)}{P'''(h)} = x_2$.

Proof. Let $y, w_1, w_2 \in (0,1)$ be given so that $w_1 < \frac{1-y}{1-x_1y}$ (as the latter quantity must be positive, w_1 exists) and $w_2 < \frac{1-y}{1-x_2y}$ (as the latter quantity must be positive, w_2 exists). The equalities in Lemma 3 arise from the following scheme of probability assignments:

$$
\begin{aligned}
P'''(h \wedge e_1 \wedge e_2) &= (x_1w_1)(x_2w_2)y \\
P'''(h \wedge e_1 \wedge \neg e_2) &= (x_1w_1)(1-x_2w_2)y \\
P'''(h \wedge \neg e_1 \wedge e_2) &= (1-x_1w_1)(x_2w_2)y \\
P'''(h \wedge \neg e_1 \wedge \neg e_2) &= (1-x_1w_1)(1-x_2w_2)y \\
P'''(\neg h \wedge e_1 \wedge e_2) &= \tfrac{(1-x_1y)(1-x_2y)w_1w_2}{1-y} \\
P'''(\neg h \wedge e_1 \wedge \neg e_2) &= (1-x_1y)w_1[1-\tfrac{(1-x_2y)w_2}{1-y}] \\
P'''(\neg h \wedge \neg e_1 \wedge e_2) &= [1-\tfrac{(1-x_1y)w_1}{1-y}](1-x_2y)w_2 \\
P'''(\neg h \wedge \neg e_1 \wedge \neg e_2) &= [1-\tfrac{(1-x_1y)w_1}{1-y}][1-\tfrac{(1-x_2y)w_2}{1-y}](1-y)
\end{aligned}
$$

Suppose there exist $x_1, x_2 \in D_m$ such that $x_1 > x_2$ and $m(x_1) \leq m(x_2)$. Then, by Lemma 3 and the definition of D_m, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P''' \in \mathcal{P}$ such that $\frac{P'''(h|e_1)}{P'''(h)} = x_1$ and $\frac{P'''(h|e_2)}{P'''(h)} = x_2$. Clearly, if the latter equalities hold, then $P'''(h|e_1) > P'''(h|e_2)$. Thus, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P''' \in \mathcal{P}$ such that $\mathcal{C}_{P'''}(h,e_1) = m(x_1) \leq m(x_2) = C_{P'''}(h,e_2)$ even if $P'''(h|e_1) > P'''(h|e_2)$, contradicting A1. Conversely, A1 implies that, for any $x_1, x_2 \in D_m$, if $x_1 > x_2$, then $m(x_1) > m(x_2)$. By a similar argument, A1 also implies that, for any $x_1, x_2 \in D_m$, if $x_1 = x_2$, then $m(x_1) = m(x_2)$. So, for A1 to hold, it must be that, for any $e,h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h|e) \leq P(h)$, then $\mathcal{C}_P(h,e) = m[\frac{P(h|e)}{P(h)}]$ and m is a strictly increasing function.

The case of confirmation ($P(h|e) > P(h)$)

Note that $P(h \wedge e) = [1 - \frac{P(\neg h|e)}{P(\neg h)} P(\neg h)]P(e)$ and $P(h) = 1 - P(\neg h)$. As a consequence, by A0, there exists a function r such that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $\mathcal{C}_P(h, e) = r[\frac{P(\neg h|e)}{P(\neg h)}, P(\neg h), P(e)]$. With no loss of generality, we will convey probabilistic coherence, regularity, and confirmation by constraining the domain of r to include triplets of values (x, y, w) such that the following conditions are jointly satisfied:

- $0 < y, w < 1$;
- $x \geq 0$, by which $x = \frac{P(\neg h|e)}{P(\neg h)} \geq 0$, so that $P(\neg h|e) \geq 0$, and thus $P(\neg h \wedge e) \geq 0$;
- $x < 1$ (conveying confirmation, i.e., $P(h|e) > P(h)$), by which $xy = P(\neg h|e) < 1$, so that $P(\neg h \wedge e) < P(e)$ and thus $P(h \wedge e) > 0$, and $xw = P(e|\neg h) < 1$, so that $P(\neg h \wedge e) < P(\neg h)$ and thus $P(\neg h \wedge \neg e) > 0$;
- $x \geq \frac{y+w-1}{yw}$ (as $y, w < 1$, the latter quantity is necessarily lower than 1), by which $xyw = P(\neg h \wedge e) \geq P(\neg h) + P(e) - 1 = y + w - 1$, and thus $P(\neg h \wedge e) + P(h \wedge e) + P(\neg h \wedge \neg e) \leq 1$.

We then posit $r : \{(x, y, w) \in [0, 1) \times (0, 1)^2 \mid x \geq \frac{y+w-1}{yw}\} \to \Re$ and denote the domain of r as D_r.

Lemma 4. For any x, y, w_1, w_2 such that $x \in [0, 1), y, w_1, w_2 \in (0, 1)$ and $x \geq \frac{y+w_1-1}{yw_1}, \frac{y+w_2-1}{yw_2}$, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P' \in \mathcal{P}$ such that $\frac{P'(\neg h|e_1)}{P'(\neg h)} = \frac{P'(\neg h|e_2)}{P'(\neg h)} = x, P'(\neg h) = y, P'(e_1) = w_1$, and $P'(e_2) = w_2$.

Proof. The equalities in Lemma 4 arise from the following scheme of probability assignments:

$$
\begin{aligned}
P'(h \wedge e_1 \wedge e_2) &= \tfrac{(1-xy)^2 w_1 w_2}{1-y} \\
P'(h \wedge e_1 \wedge \neg e_2) &= (1-xy)w_1[1 - \tfrac{(1-xy)w_2}{1-y}] \\
P'(h \wedge \neg e_1 \wedge e_2) &= [1 - \tfrac{(1-xy)w_1}{1-y}](1-xy)w_2 \\
P'(h \wedge \neg e_1 \wedge \neg e_2) &= [1 - \tfrac{(1-xy)w_1}{1-y}][1 - \tfrac{(1-xy)w_2}{1-y}](1-y) \\
P'(\neg h \wedge e_1 \wedge e_2) &= (xw_1)(xw_2)y \\
P'(\neg h \wedge e_1 \wedge \neg e_2) &= (xw_1)(1-xw_2)y \\
P'(\neg h \wedge \neg e_1 \wedge e_2) &= (1-xw_1)(xw_2)y \\
P'(\neg h \wedge \neg e_1 \wedge \neg e_2) &= (1-xw_1)(1-xw_2)y
\end{aligned}
$$

Suppose there exist $(x, y, w_1), (x, y, w_2) \in D_r$ such that $r(x, y, w_1) \neq r(x, y, w_2)$. Then, by Lemma 4 and the definition of D_r, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P' \in \mathcal{P}$ such that $\frac{P'(\neg h|e_1)}{P'(\neg h)} = \frac{P'(\neg h|e_2)}{P'(\neg h)} = x, P'(\neg h) = y, P'(e_1) = w_1$, and $P'(e_2) = w_2$. By the probability calculus, if the latter equalities hold, then $P'(h|e_1) = P'(h|e_2)$. Thus, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P' \in \mathcal{P}$ such that $\mathcal{C}_{P'}(h, e_1) = r(x, y, w_1) \neq r(x, y, w_2) = C_{P'}(h, e_2)$ even if $P'(h|e_1) = P'(h|e_2)$, contradicting A1. Conversely, A1 implies that, for any $(x, y, w_1), (x, y, w_2) \in D_r$, $r(x, y, w_1) = r(x, y, w_2)$. So, for A1 to hold, there must exist a function s such that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h|e) > P(h)$, then $\mathcal{C}_P(h, e) = s[\frac{P(\neg h|e)}{P(\neg h)}, P(\neg h)]$ and $s(x, y) = r(x, y, w)$. We then posit $s : \{(x, y) \in [0, 1) \times (0, 1)\} \to \Re$ and denote the domain of s as D_s.

Lemma 5. For any x, y_1, y_2 such that $x \in [0, 1), y_1, y_2 \in (0, 1)$, there exist $e, h_1, h_2 \in \mathcal{L}_c$ and $P'' \in \mathcal{P}$ such that $\frac{P''(\neg h_1|e)}{P''(\neg h_1)} = \frac{P''(\neg h_2|e)}{P'(\neg h_2)} = x, P''(\neg h_1) = y_1$, and $P''(\neg h_2) = y_2$.

Proof. Let $w \in (0, 1)$ be given so that $w \leq \frac{1-y_1}{1-xy_1}, \frac{1-y_2}{1-xy_2}$ (as the latter quantities must all be positive, w exists). The equalities in Lemma 5 arise from the following scheme of probability assignments:

$$
\begin{aligned}
P''(h_1 \wedge h_2 \wedge e) &= (1 - xy_1)(1 - xy_2)w \\
P''(h_1 \wedge h_2 \wedge \neg e) &= [1 - \tfrac{(1-xw)y_1}{1-w}][1 - \tfrac{(1-xw)y_2}{1-w}](1 - w) \\
P''(h_1 \wedge \neg h_2 \wedge e) &= (1 - xy_1)(xy_2)w \\
P''(h_1 \wedge \neg h_2 \wedge \neg e) &= [1 - \tfrac{(1-xw)y_1}{1-w}](1 - xw)y_2 \\
P''(\neg h_1 \wedge h_2 \wedge e) &= (xy_1)(1 - xy_2)w \\
P''(\neg h_1 \wedge h_2 \wedge \neg e) &= (1 - xw)y_1[1 - \tfrac{(1-xw)y_2}{1-w}] \\
P''(\neg h_1 \wedge \neg h_2 \wedge e) &= (xy_1)(xy_2)w \\
P''(\neg h_1 \wedge \neg h_2 \wedge \neg e) &= \tfrac{(1-xw)^2 y_1 y_2}{1-w}
\end{aligned}
$$

Suppose there exist $(x, y_1), (x, y_2) \in D_s$ such that $s(x, y_1) \neq s(x, y_2)$. Then, by Lemma 5 and the definition of D_s, there exist $e, h_1, h_2 \in \mathcal{L}_c$ and $P'' \in \mathcal{P}$ such that $\frac{P''(\neg h_1|e)}{P''(\neg h_1)} = \frac{P''(\neg h_2|e)}{P''(\neg h_2)} = x$, $P''(\neg h_1) = y_1$, $P''(\neg h_2) = y_2$, and $P''(e) = w$. If the latter equalities hold, then $\mathcal{C}_{P''}(\neg h_1, e) = m[\frac{P''(\neg h_1|e)}{P''(\neg h_1)}] = m[\frac{P''(\neg h_2|e)}{P''(\neg h_2)}] = C_{P''}(\neg h_2, e)$. Thus, there exist $e, h_1, h_2 \in \mathcal{L}_c$ and $P'' \in \mathcal{P}$

such that $\mathcal{C}_{P''}(h_1, e) = s(x, y_1) \neq s(x, y_2) = \mathcal{C}_{P''}(h_2, e)$ even if $\mathcal{C}_{P''}(\neg h_1, e) = \mathcal{C}_{P''}(\neg h_2, e)$, contradicting A3. Conversely, A3 implies that, for any $(x, y_1), (x, y_2) \in D_s$, $s(x, y_1) = s(x, y_2)$. So, for A3 to hold, there must exist a function t such that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h|e) > P(h)$, then $\mathcal{C}_P(h, e) = t[\frac{P(\neg h|e)}{P(\neg h)}]$ and $t(x) = s(x, y)$. We then posit $t : [0, 1) \to \Re$ and denote the domain of t as D_t.

Lemma 6. For any $x_1, x_2 \in [0, 1)$, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P''' \in \mathcal{P}$ such that $\frac{P'''(\neg h|e_1)}{P'''(\neg h)} = x_1$ and $\frac{P'''(\neg h|e_2)}{P'''(\neg h)} = x_2$.

Proof. Let $y, w_1, w_2 \in (0, 1)$ be given so that $w_1 < \frac{1-y}{1-x_1 y}$ (as the latter quantity must be positive, w_1 exists) and $w_2 < \frac{1-y}{1-x_2 y}$ (as the latter quantity must be positive, w_2 exists). The equalities in Lemma 6 arise from the following scheme of probability assignments:

$$\begin{aligned}
P'''(h \wedge e_1 \wedge e_2) &= \tfrac{(1-x_1 y)(1-x_2 y)w_1 w_2}{1-y} \\
P'''(h \wedge e_1 \wedge \neg e_2) &= (1 - x_1 y)w_1[1 - \tfrac{(1-x_2 y)w_2}{1-y}] \\
P'''(h \wedge \neg e_1 \wedge e_2) &= [1 - \tfrac{(1-x_1 y)w_1}{1-y}](1 - x_2 y)w_2 \\
P'''(h \wedge \neg e_1 \wedge \neg e_2) &= [1 - \tfrac{(1-x_1 y)w_1}{1-y}][1 - \tfrac{(1-x_2 y)w_2}{1-y}](1 - y) \\
P'''(\neg h \wedge e_1 \wedge e_2) &= (x_1 w_1)(x_2 w_2)y \\
P'''(\neg h \wedge e_1 \wedge \neg e_2) &= (x_1 w_1)(1 - x_2 w_2)y \\
P'''(\neg h \wedge \neg e_1 \wedge e_2) &= (1 - x_1 w_1)(x_2 w_2)y \\
P'''(\neg h \wedge \neg e_1 \wedge \neg e_2) &= (1 - x_1 w_1)(1 - x_2 w_2)y
\end{aligned}$$

Suppose there exist $x_1, x_2 \in D_t$ such that $x_1 > x_2$ and $t(x_1) \geq t(x_2)$. Then, by Lemma 6 and the definition of D_t, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P''' \in \mathcal{P}$ such that $\frac{P'''(\neg h|e_1)}{P'''(\neg h)} = x_1$ and $\frac{P'''(\neg h|e_2)}{P'''(\neg h)} = x_2$. By the probability calculus, if the latter equalities hold, then $P'''(h|e_1) < P'''(h|e_2)$. Thus, there exist $e_1, e_2, h \in \mathcal{L}_c$ and $P''' \in \mathcal{P}$ such that $\mathcal{C}_{P'''}(h, e_1) = t(x_1) \geq t(x_2) = C_{P'''}(h, e_2)$ even if $P'''(h|e_1) < P'''(h|e_2)$, contradicting A1. Conversely, A1 implies that, for any $x_1, x_2 \in D_t$, if $x_1 > x_2$, then $t(x_1) < t(x_2)$. By a similar argument, A1 also implies that, for any $x_1, x_2 \in D_t$, if $x_1 = x_2$, then $t(x_1) = t(x_2)$. So, for A1 to hold, it must be that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, if $P(h|e) > P(h)$, then $\mathcal{C}_P(h, e) = t[\frac{P(\neg h|e)}{P(\neg h)}]$ and t is a strictly decreasing function.

Summing up, if A0-A3 hold, then for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, (i) in case $P(h|e) \leq P(h)$, $\mathcal{C}_P(h,e) = m[\frac{P(h|e)}{P(h)}]$ and m is a strictly increasing function, thus $\mathcal{C}_P(h,e_1)$ is a strictly increasing function of $z(h,e)$, and (ii) in case $P(h|e) > P(h)$, $\mathcal{C}_P(h,e) = t[\frac{P(\neg h|e)}{P(\neg h)}]$ and t is a strictly decreasing function, thus $\mathcal{C}_P(h,e)$ is again a strictly increasing function of $z(h,e)$. As (i)-(ii) are exhaustive, for A0-A3 to hold, it must be that, for any $e, h \in \mathcal{L}_c$ and any $P \in \mathcal{P}$, $\mathcal{C}_P(h,e) = f[z(h,e)]$ and f is a strictly increasing function.